Complexity Theory

Part Two

Recap from Last Time

The Complexity Class **P**

- The ***complexity class* P** (for ***p***olynomial time) contains all problems that can be solved in polynomial time.
- Formally:

$$\mathbf{P} = \{ \, L \mid \text{There is a polynomial-time decider for } L \, \}$$

The Complexity Class **NP**

- The complexity class **NP** (***nondeterministic polynomial time***) contains all problems that can be verified in polynomial time.
- Formally:

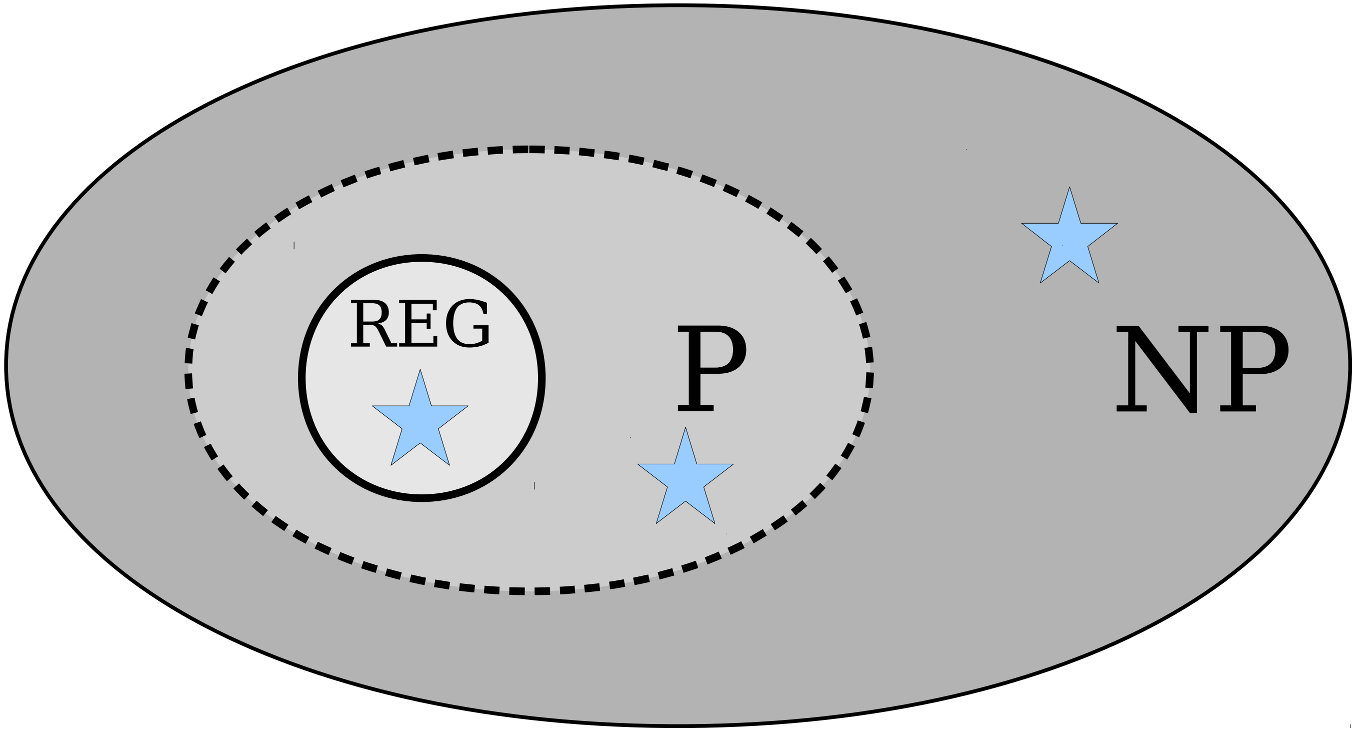
  NP = { L | There is a polynomial-time verifier for L }

- This means a verifier V's runtime is a polynomial in $|w|$ (that is, V's runtime is $O(|w|^k)$ for some integer k).

So how *are* we going to reason about **P** and **NP**?

New Stuff!

A Challenge



Problems in **NP** vary widely in their difficulty, even if **P** = **NP**.

How can we rank the relative difficulties of problems?

Reducibility

Maximum Matching

- Given an undirected graph G, a ***matching*** in G is a set of edges such that no two edges share an endpoint.
- A ***maximum matching*** is a matching with the largest number of edges.

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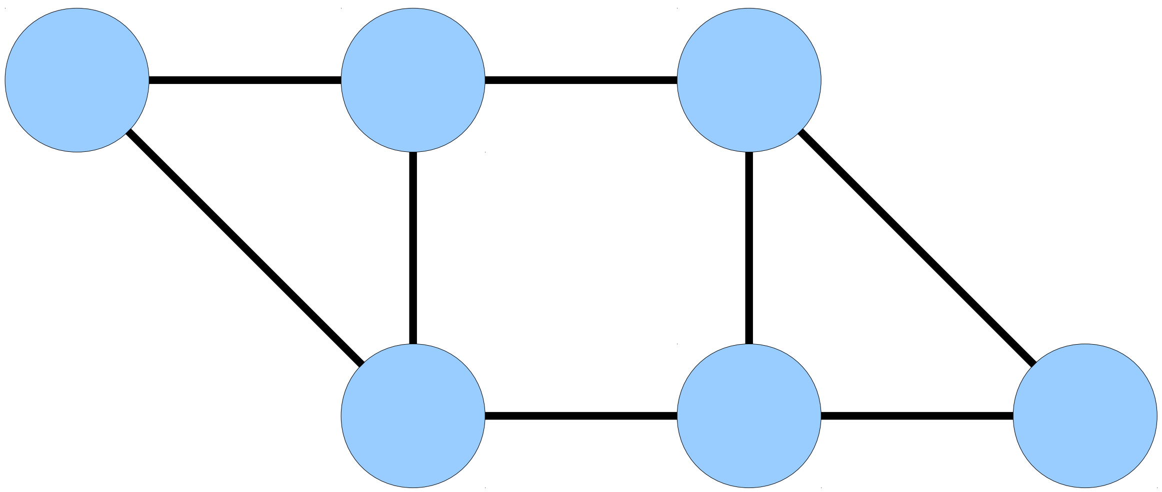

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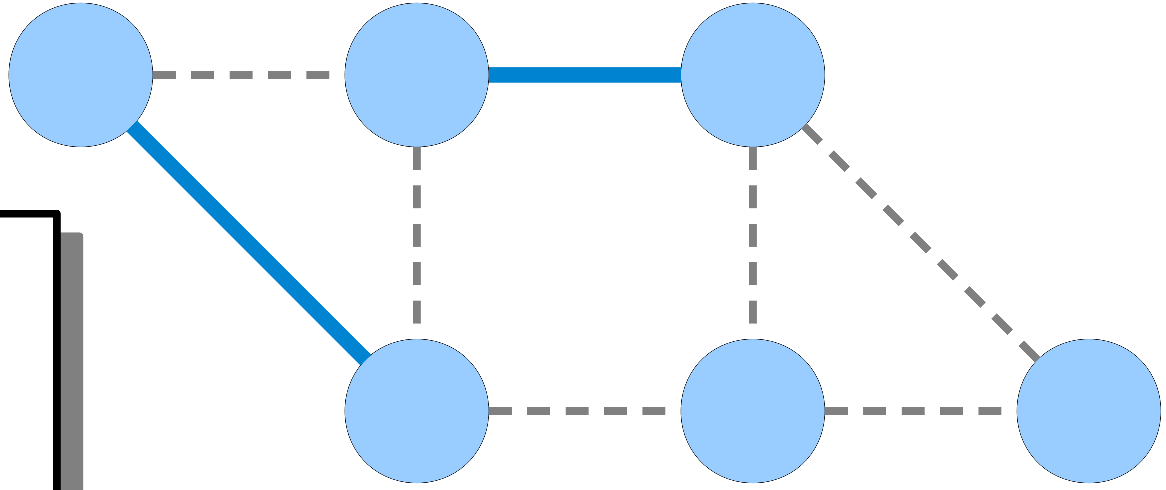
A matching, but not a maximum matching.



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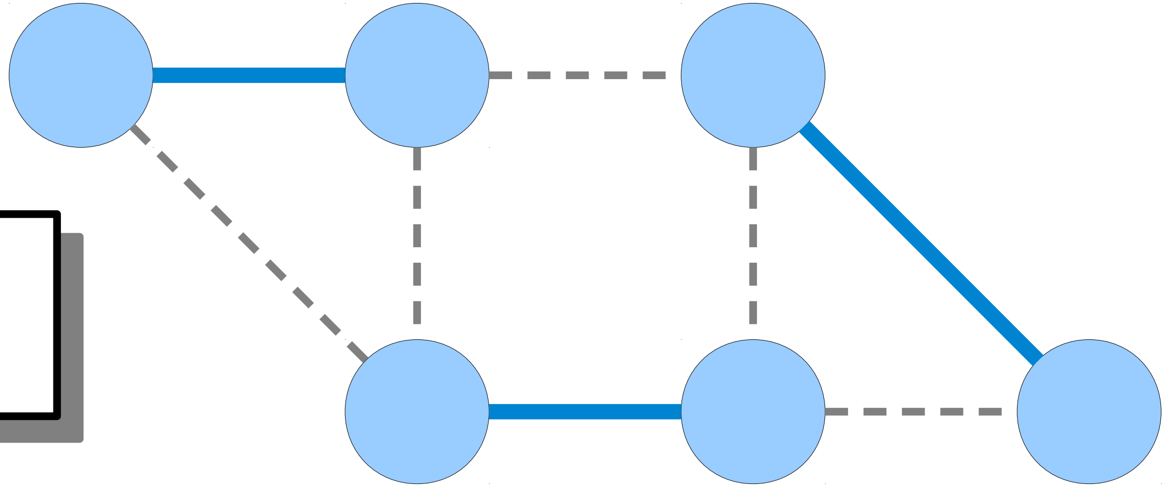

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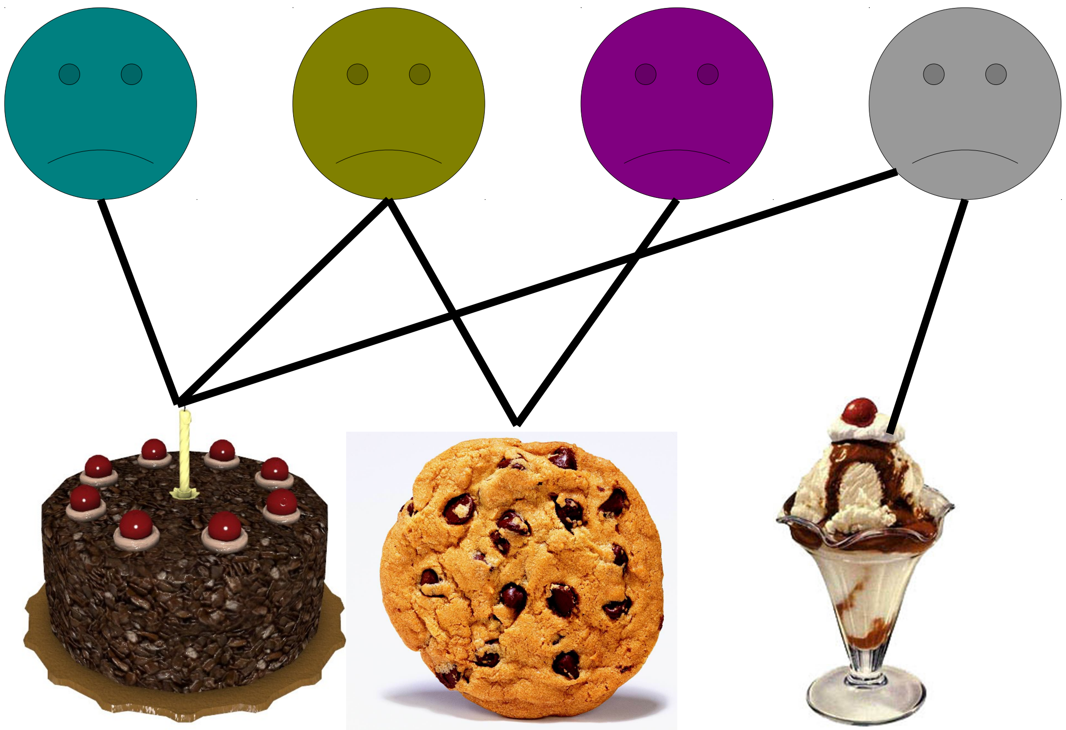

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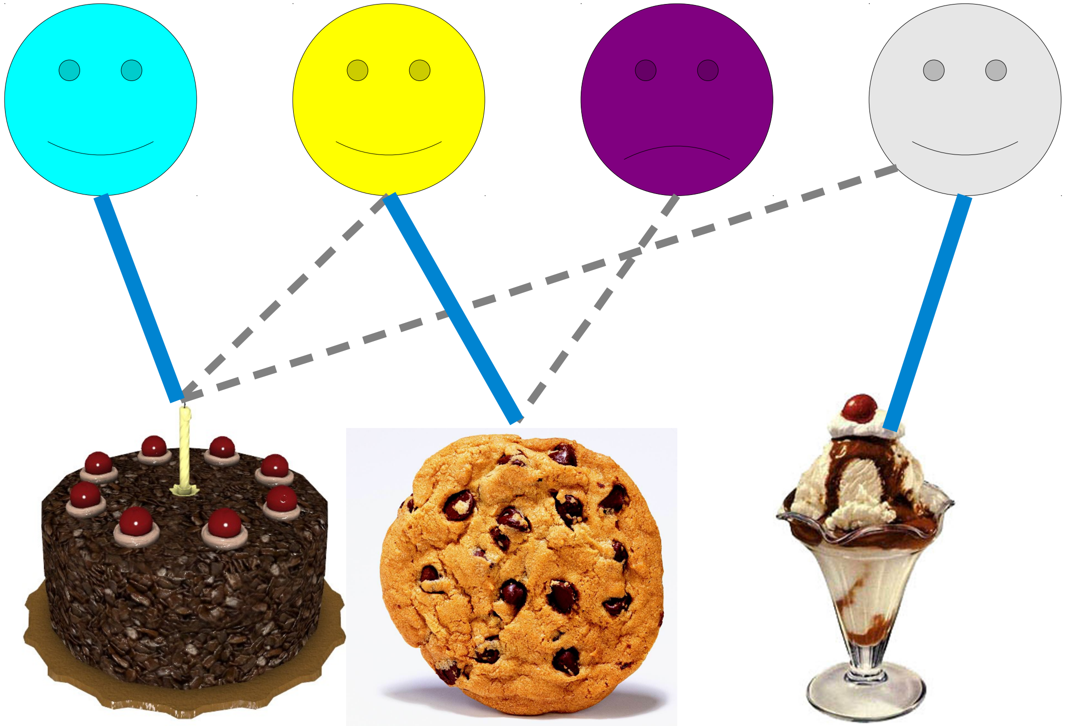
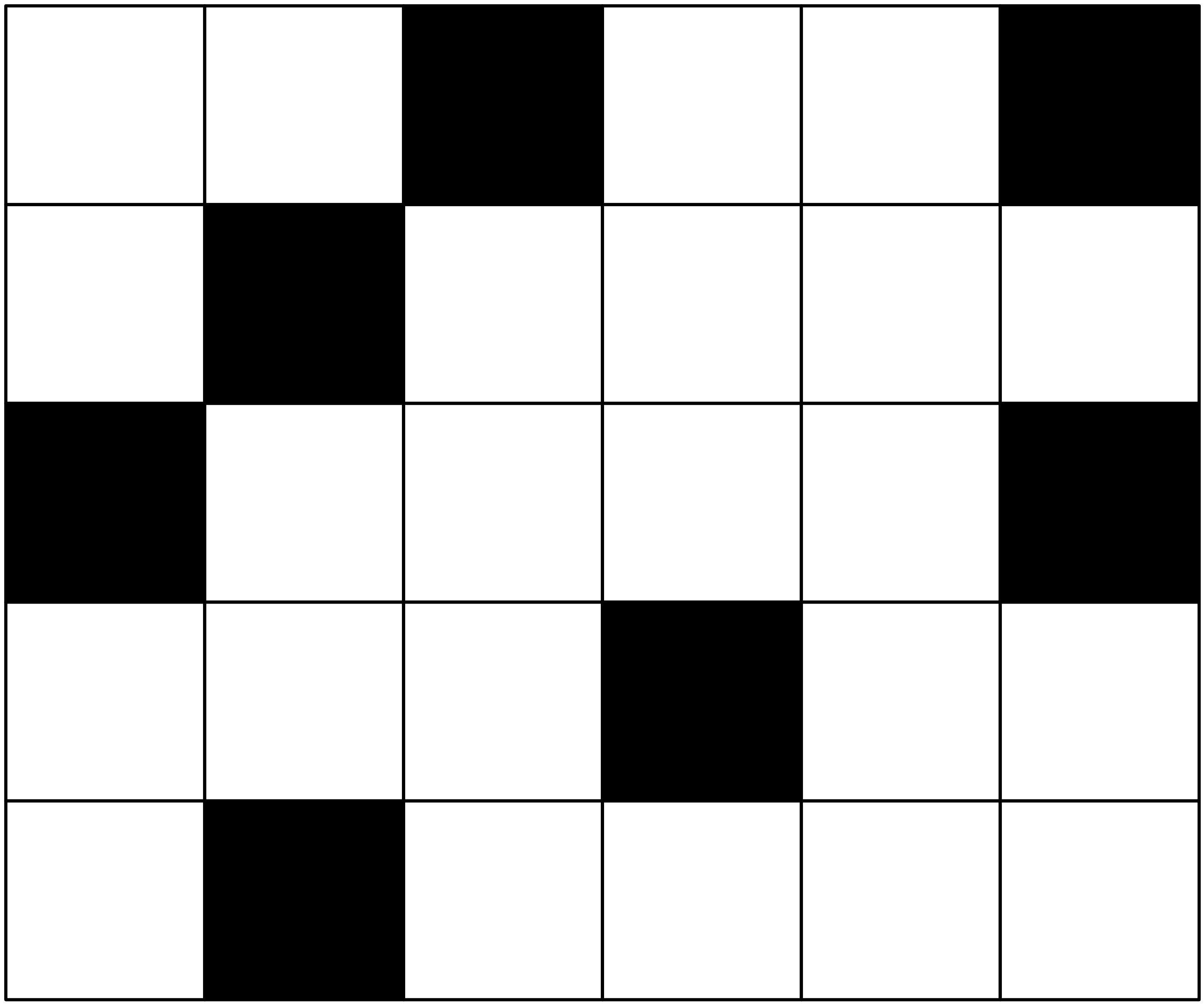


Maximum Matching

- Jack Edmonds' paper “Paths, Trees, and Flowers” gives a polynomial-time algorithm for finding maximum matchings.
    - (This is the same Edmonds as in “Cobham-Edmonds Thesis.”)
- Using this fact, what other problems can we solve?

Domino Tiling



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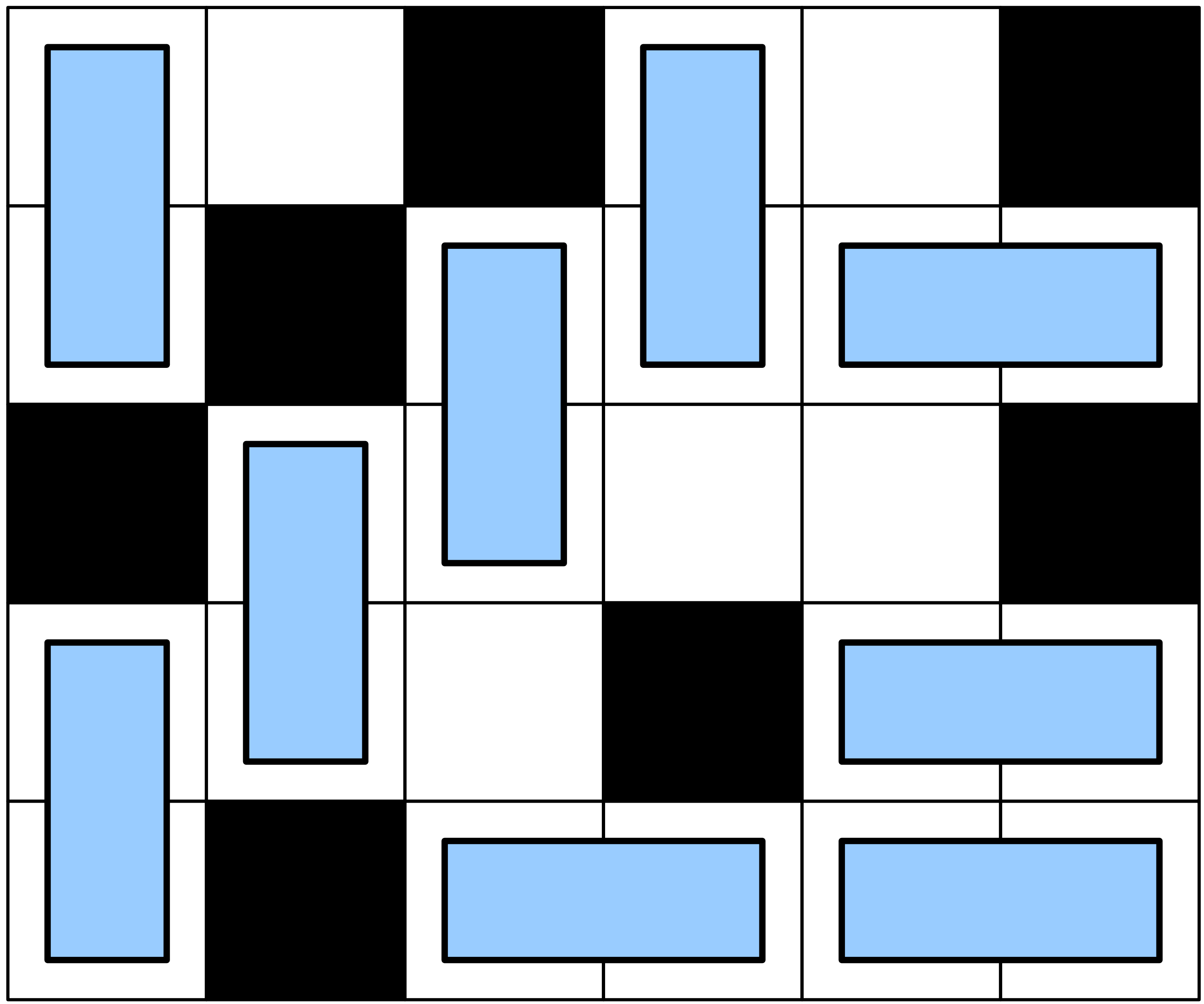

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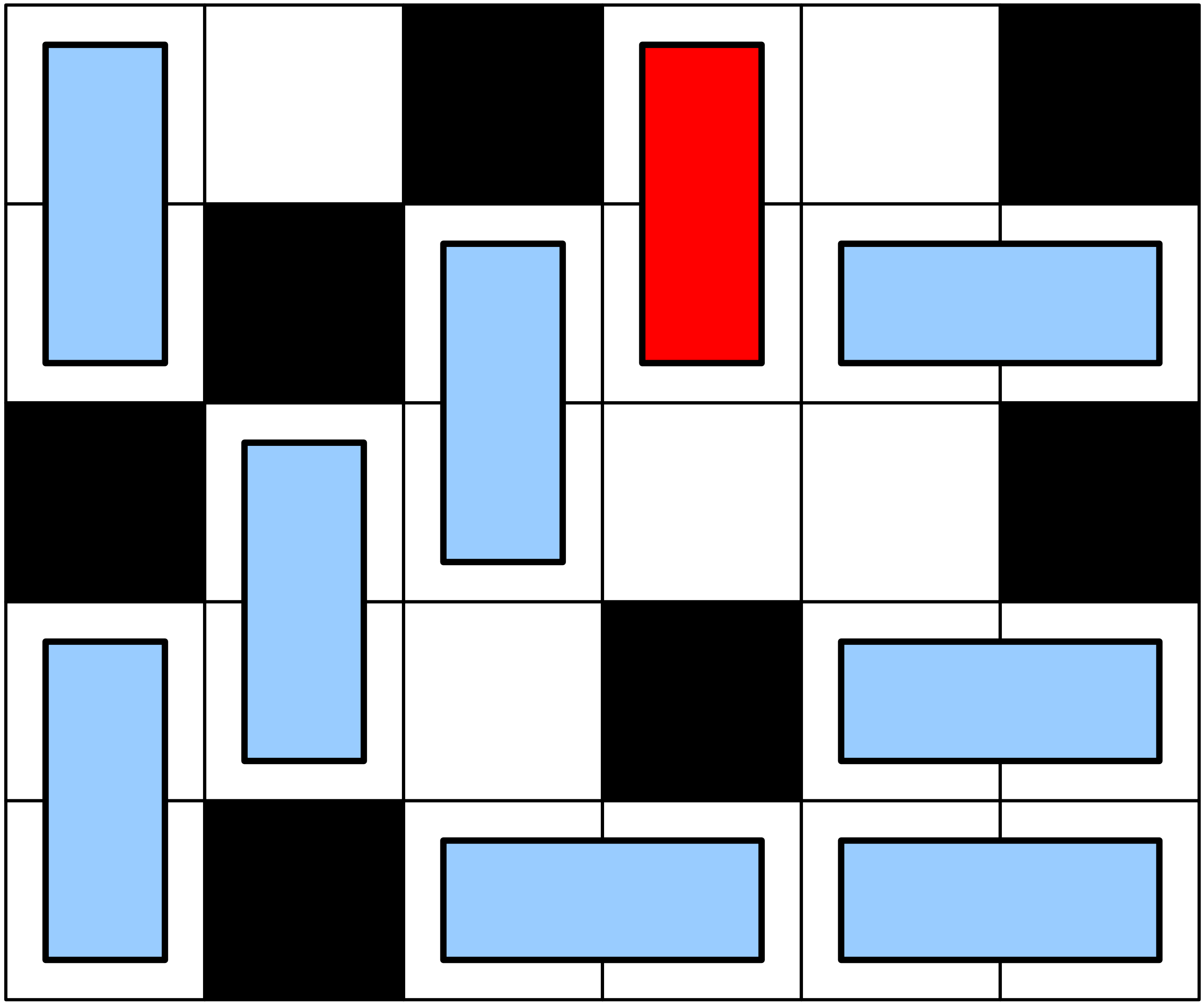

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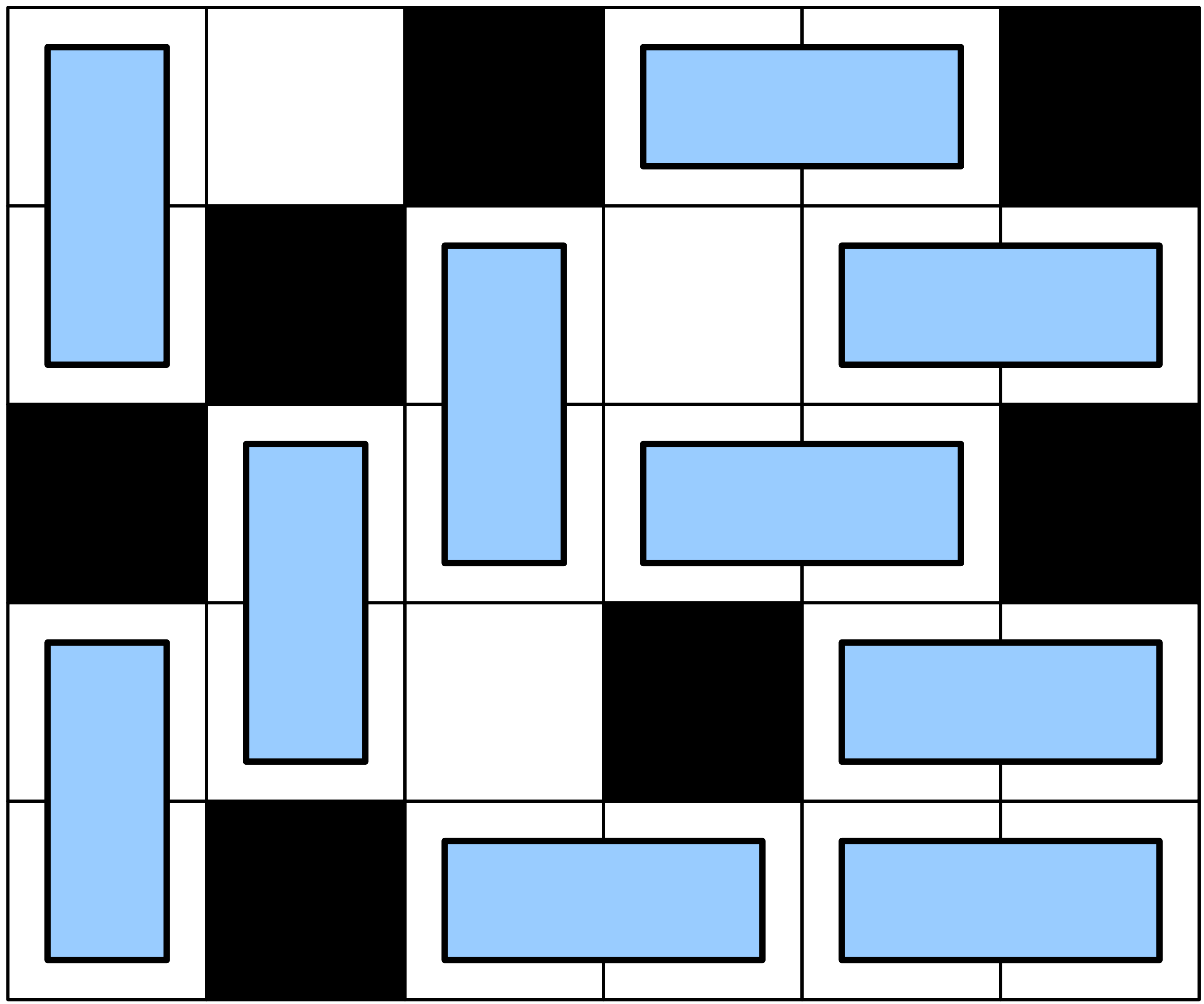

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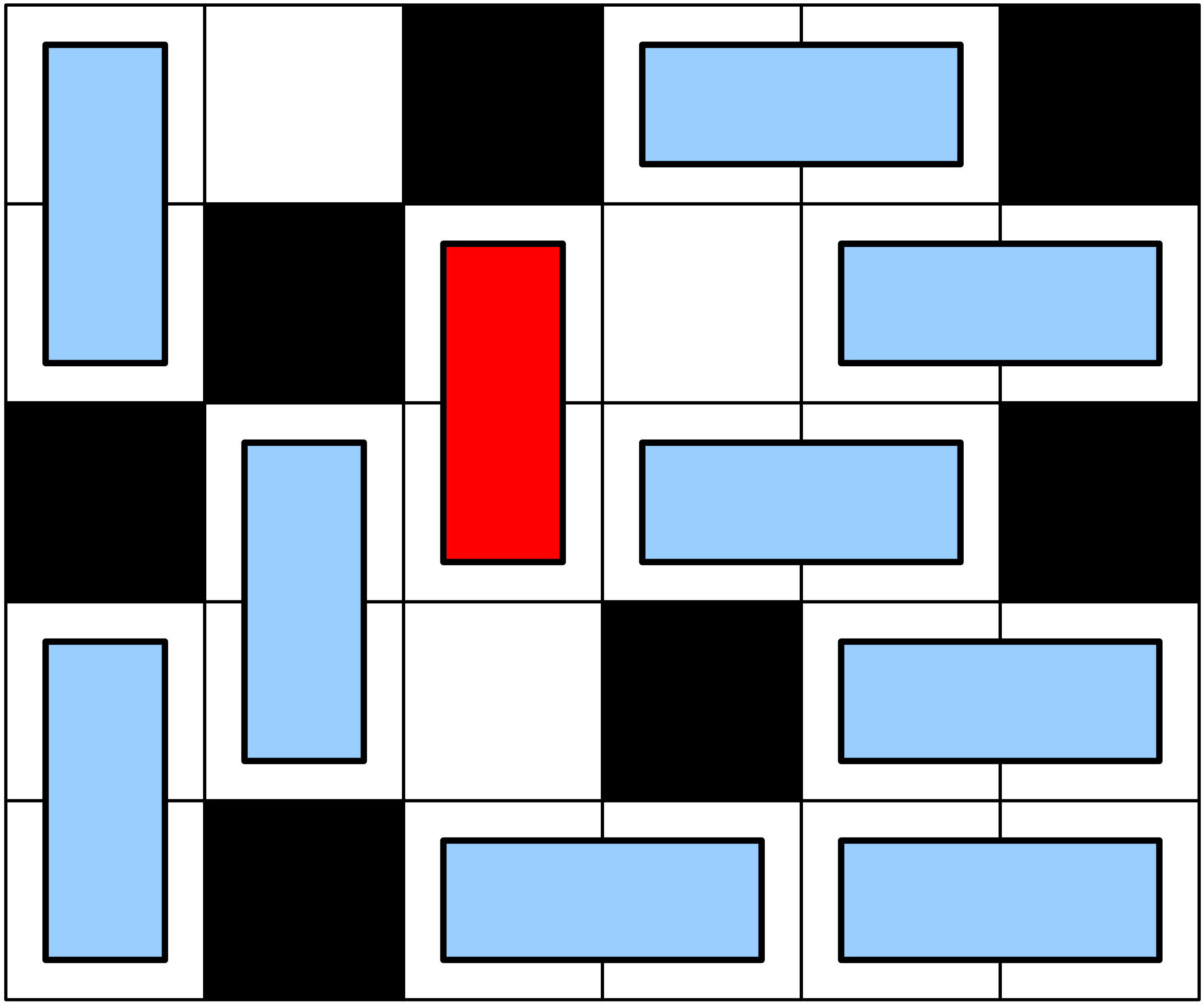

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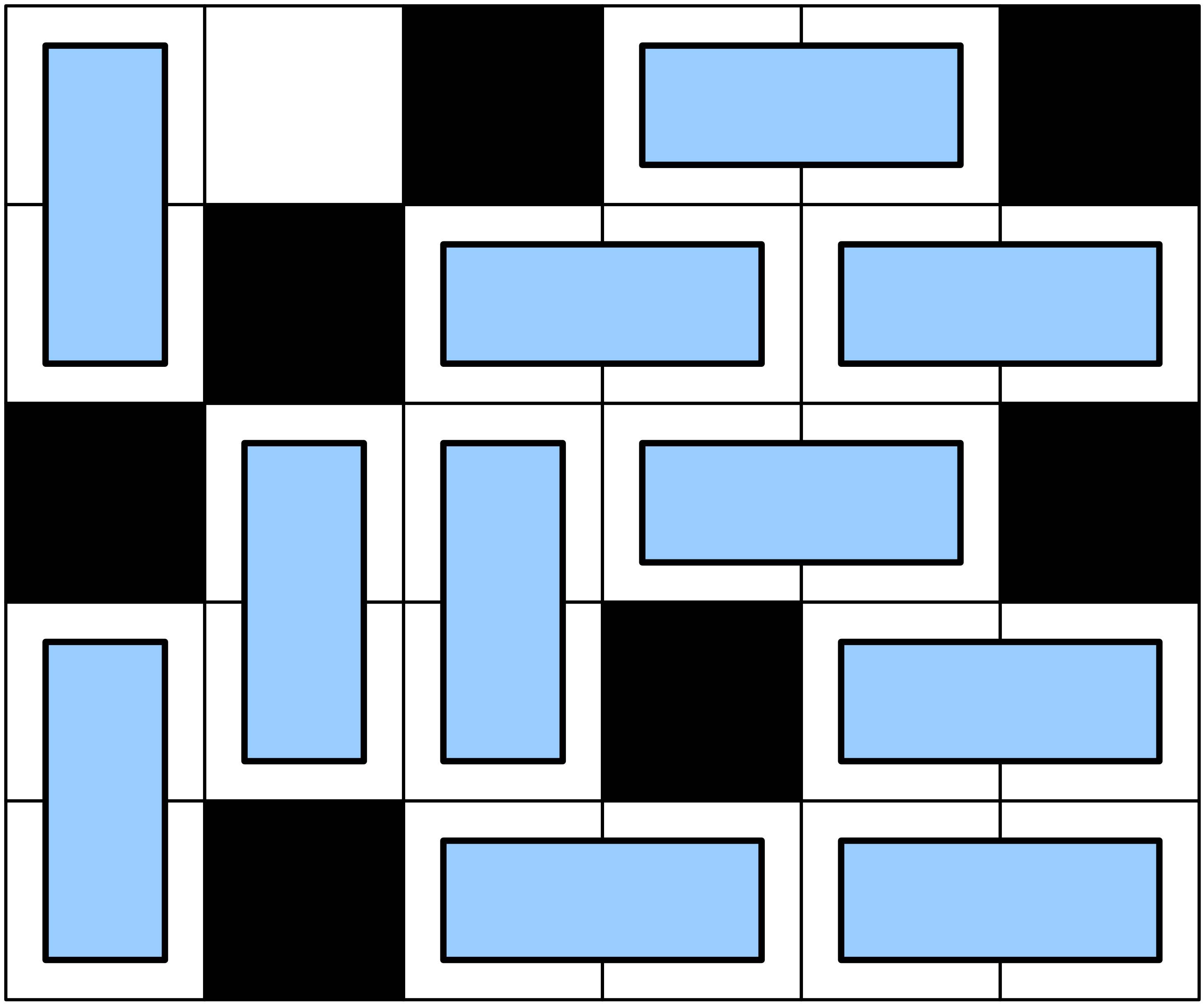
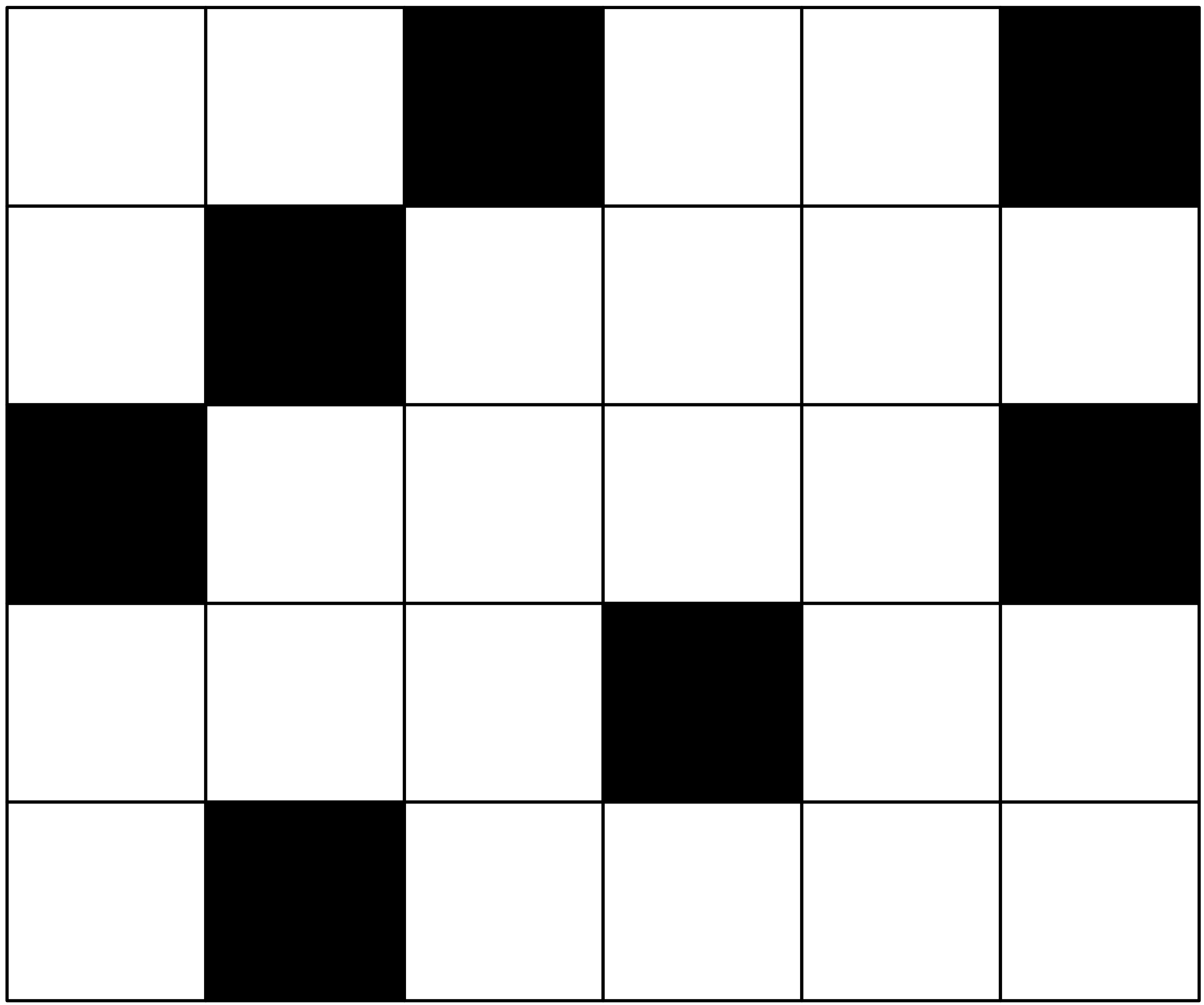


Solving Domino Tiling



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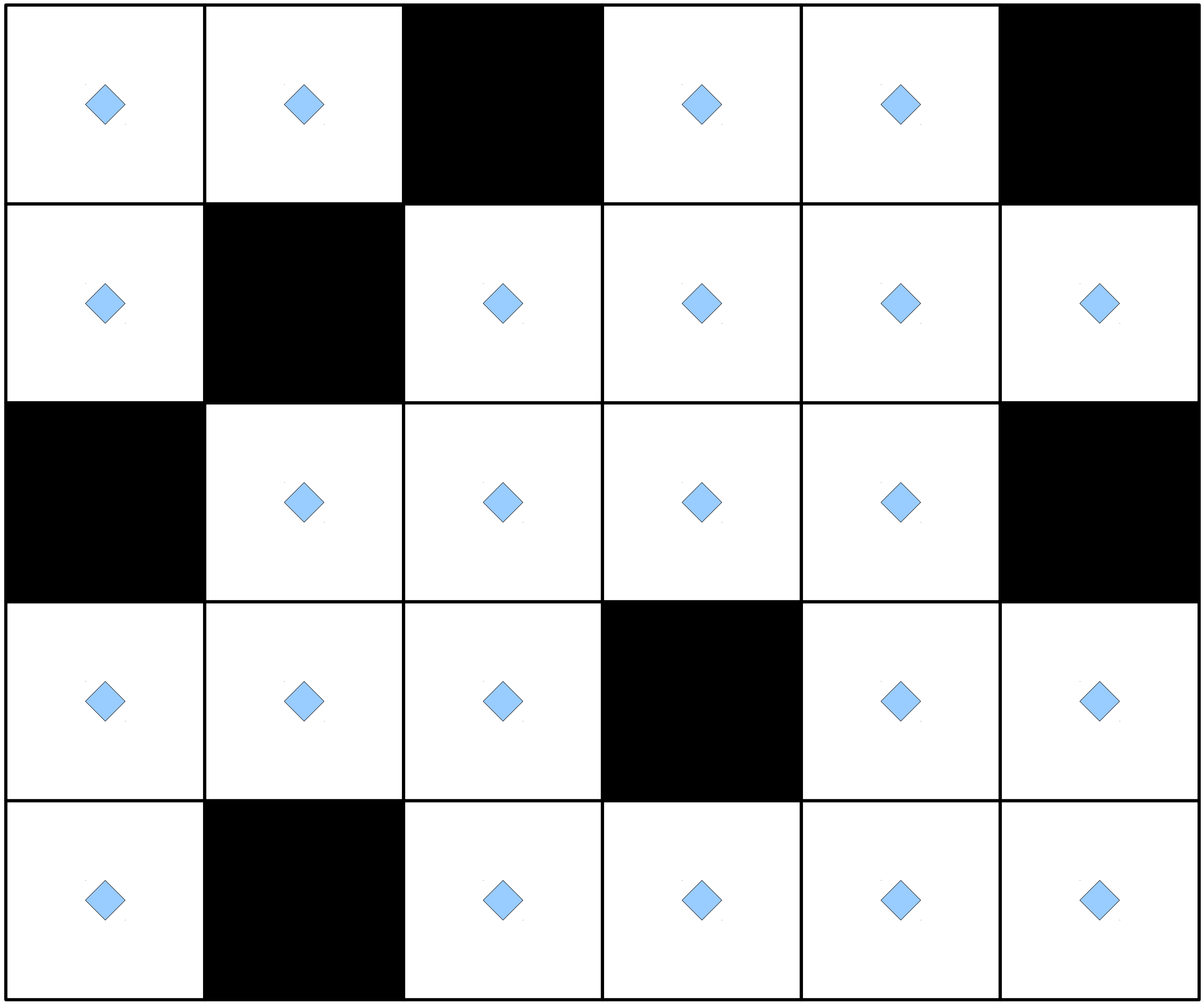

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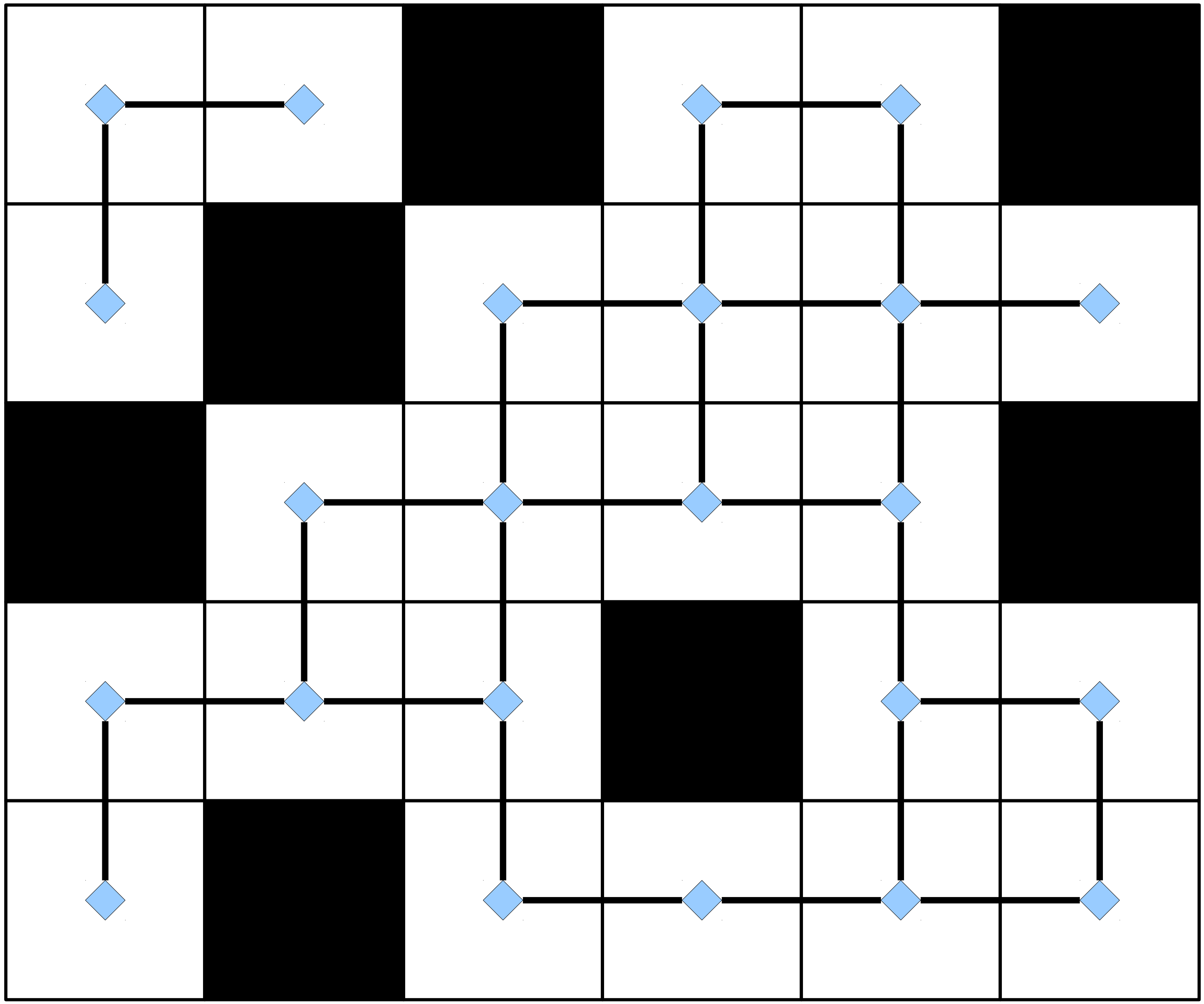

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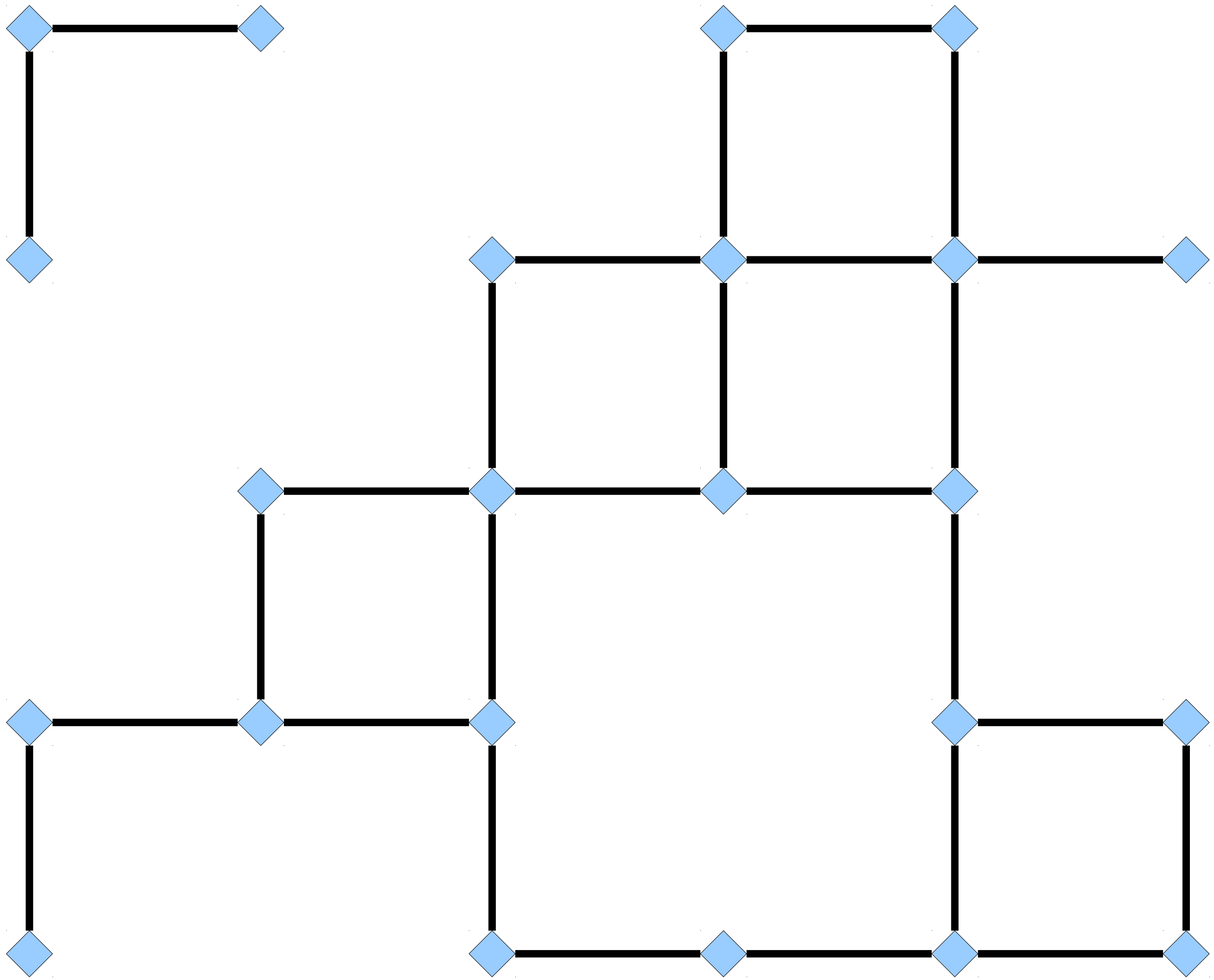

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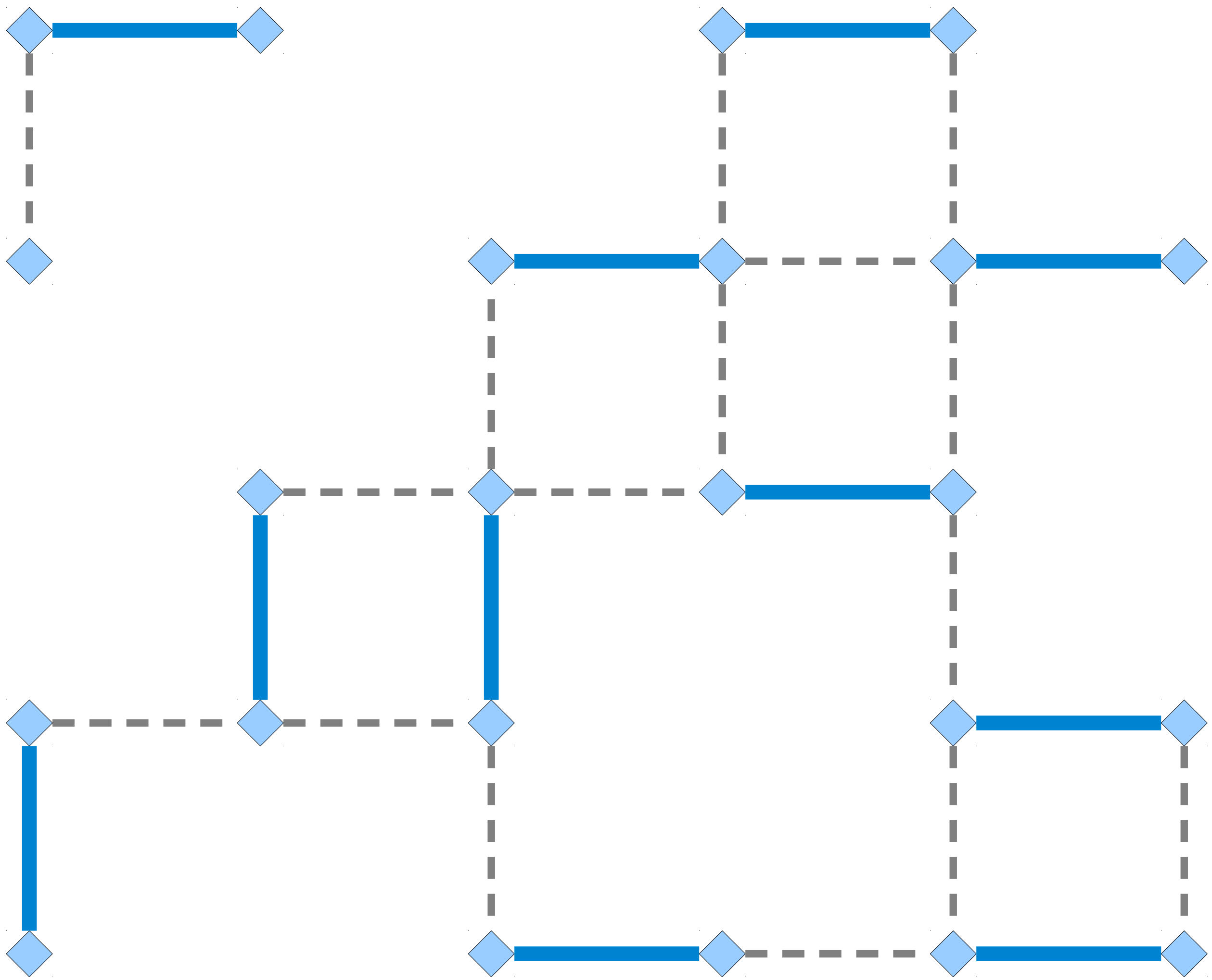

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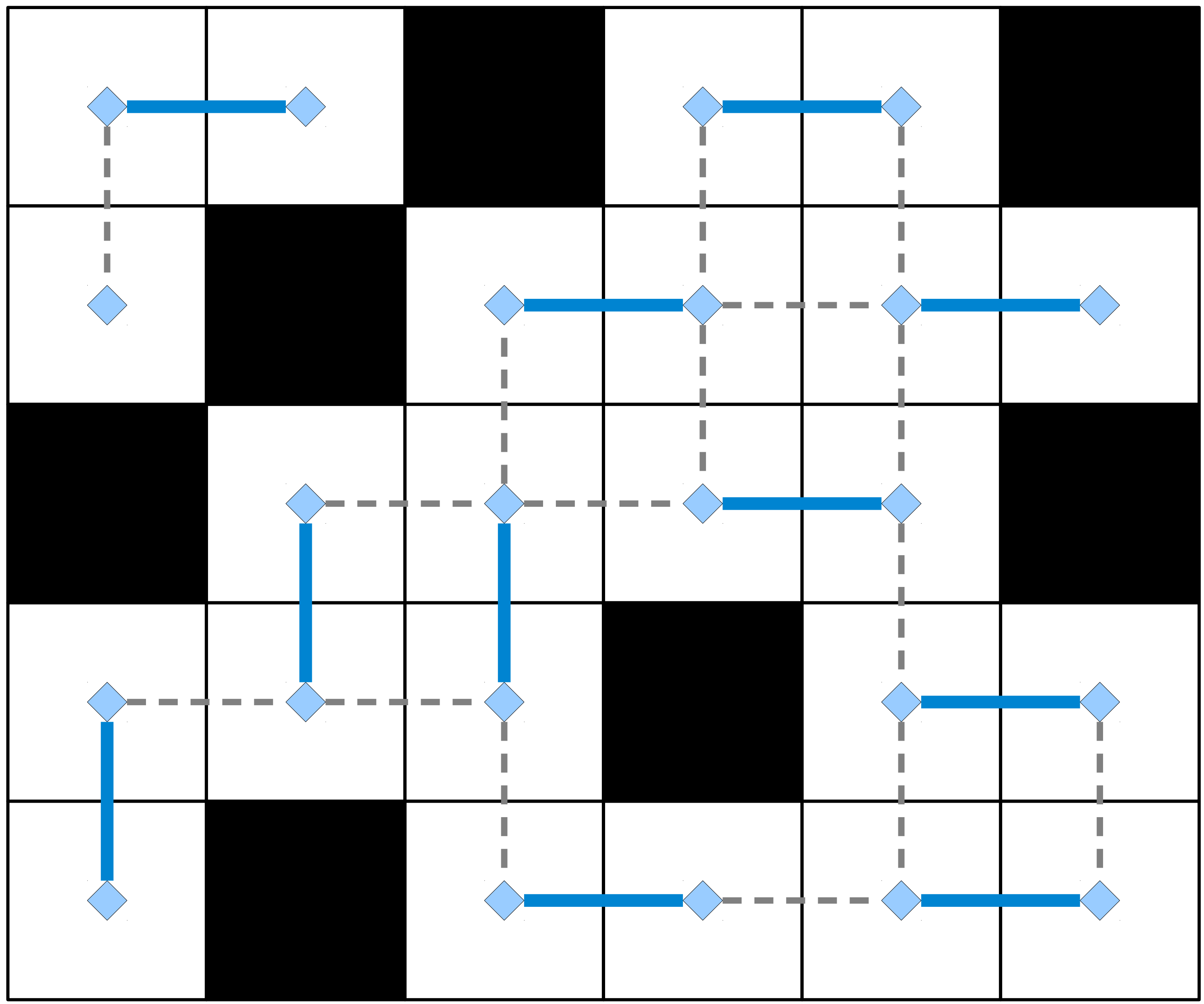

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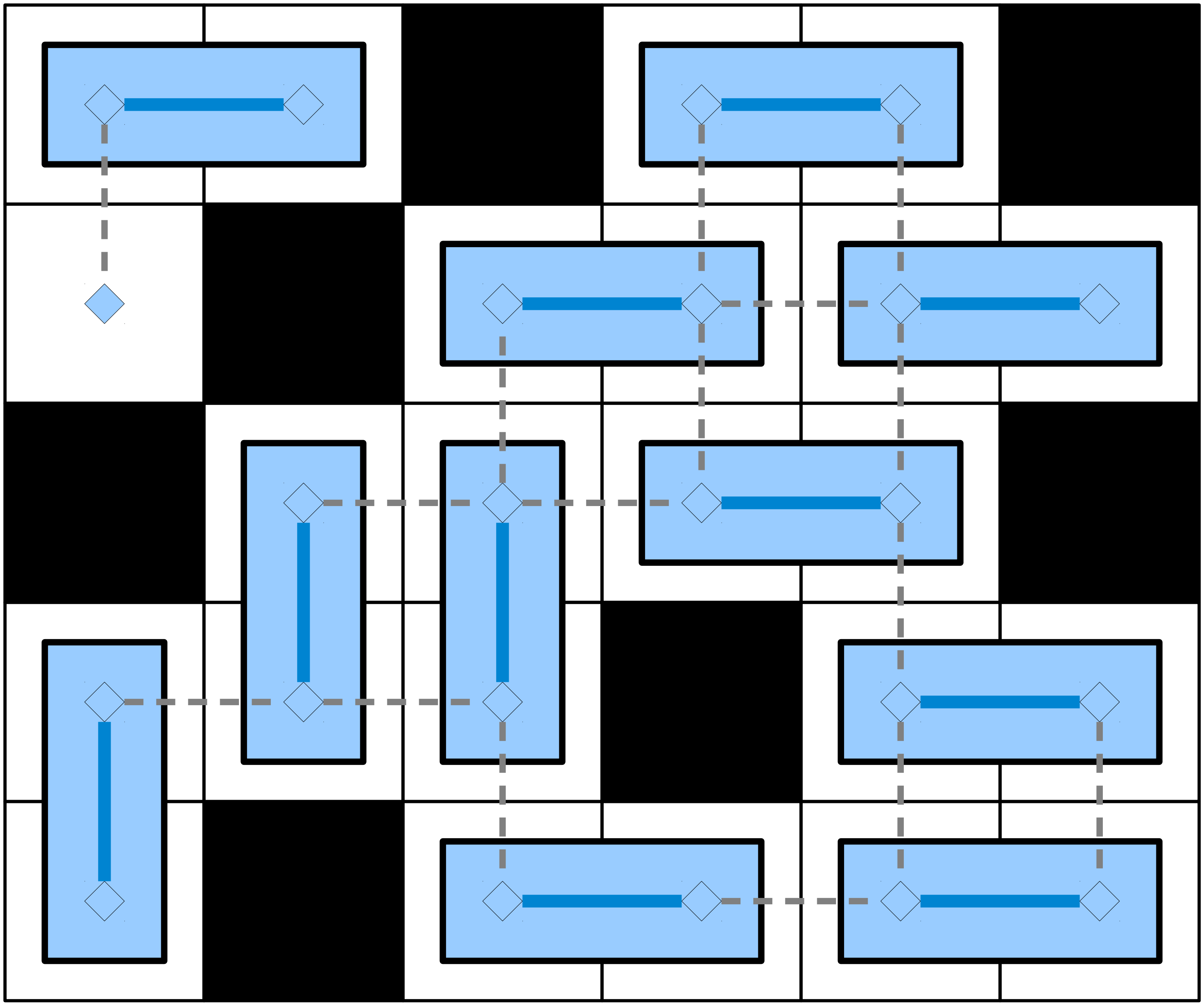

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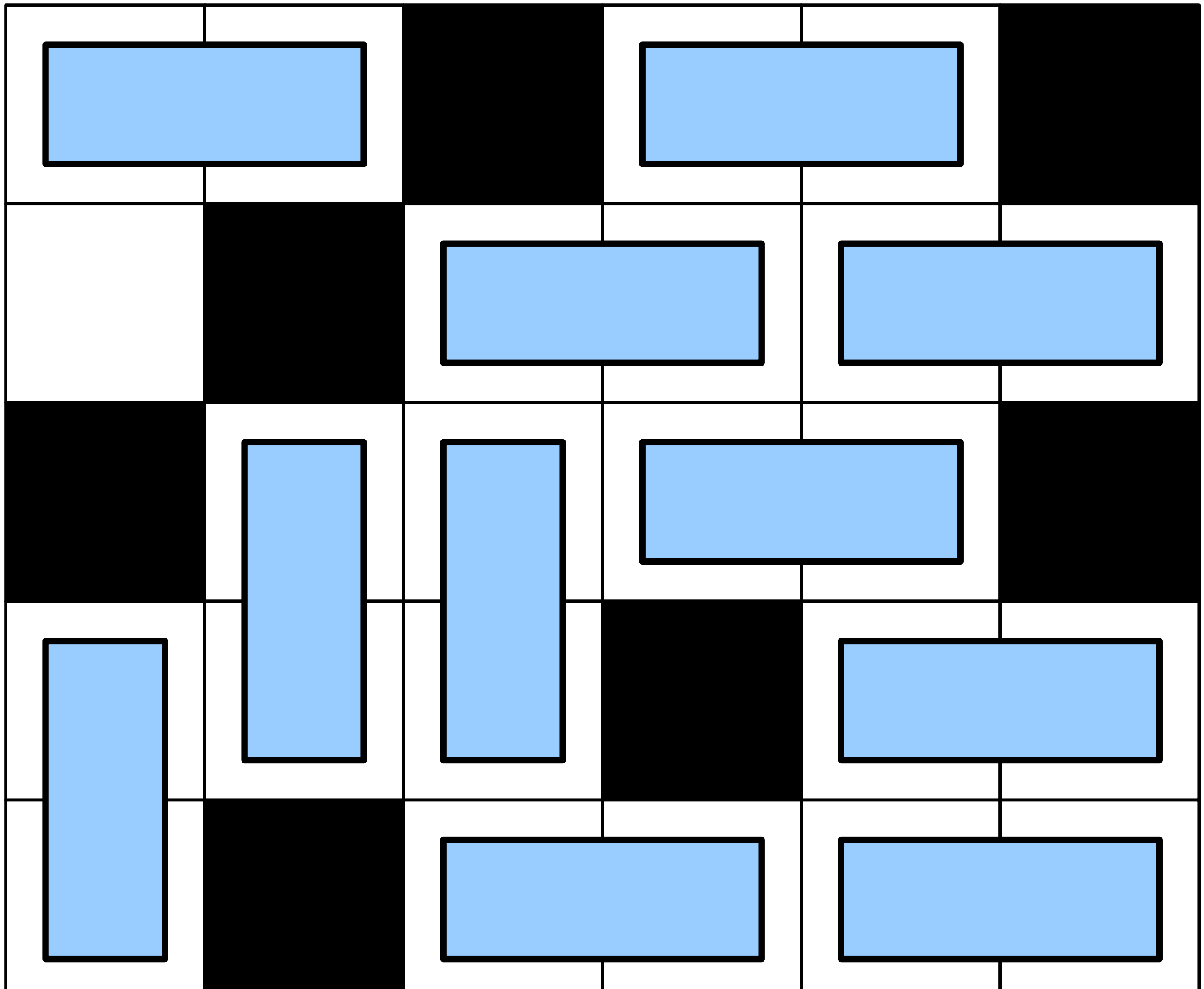

In Pseudocode

```
boolean canPlaceDominos(Grid G, int k) {
  return hasMatching(gridToGraph(G), k);
}
```

Based on this connection between maximum matching and domino tiling, which of the following statements would be more proper to conclude?

A. Finding a maximum matching isn't any more difficult (in BigO/P-NP terms) than tiling a grid with dominoes.

B. Tiling a grid with dominoes isn't any more difficult (in BigO/P-NP terms) than finding a maximum matching.

Answer at **PollEv.com/cs103** or text **CS103** to **22333** once to join, then ***A*** or ***B***.

Intuition:

Tiling a grid with dominoes can't be “harder” than solving maximum matching, because if we can solve maximum matching efficiently, we can solve domino tiling efficiently.

Another Example

Reachability

- Consider the following problem:

  Given an directed graph $\boldsymbol{G}$ ***and nodes*** $\boldsymbol{s}$ ***and*** $\boldsymbol{t}$ ***in*** $\boldsymbol{G}$***, is there a path from*** $\boldsymbol{s}$ ***to*** $\boldsymbol{t}$***?***

- It's known that this problem can be solved in polynomial time (use DFS or BFS).
- Given that we can solve the reachability problem in polynomial time, what other problems can we solve in polynomial time?

Converter Conundrums

- Suppose that you want to plug your laptop into a projector.
- Your laptop only has a VGA output, but the projector needs HDMI input.
- You have a box of connectors that convert various types of input into various types of output (for example, VGA to DVI, DVI to DisplayPort, etc.)
- ***Question:*** Can you plug your laptop into the projector?

Converter Conundrums

Connectors

RGB to USB
VGA to DisplayPort
DB13W3 to CATV
DisplayPort to RGB
DB13W3 to HDMI
DVI to DB13W3
S-Video to DVI
FireWire to SDI
VGA to RGB
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USB to S-Video
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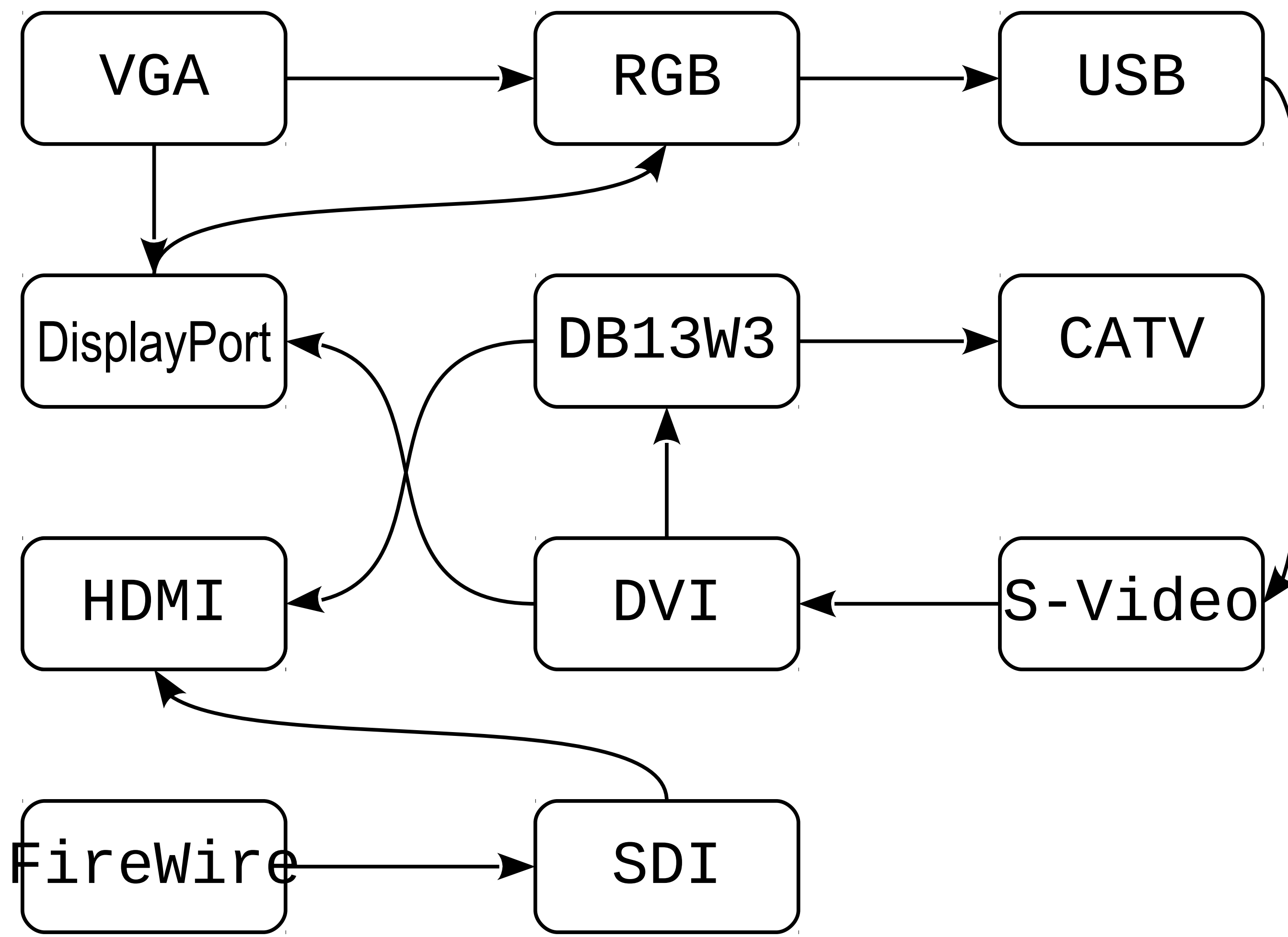

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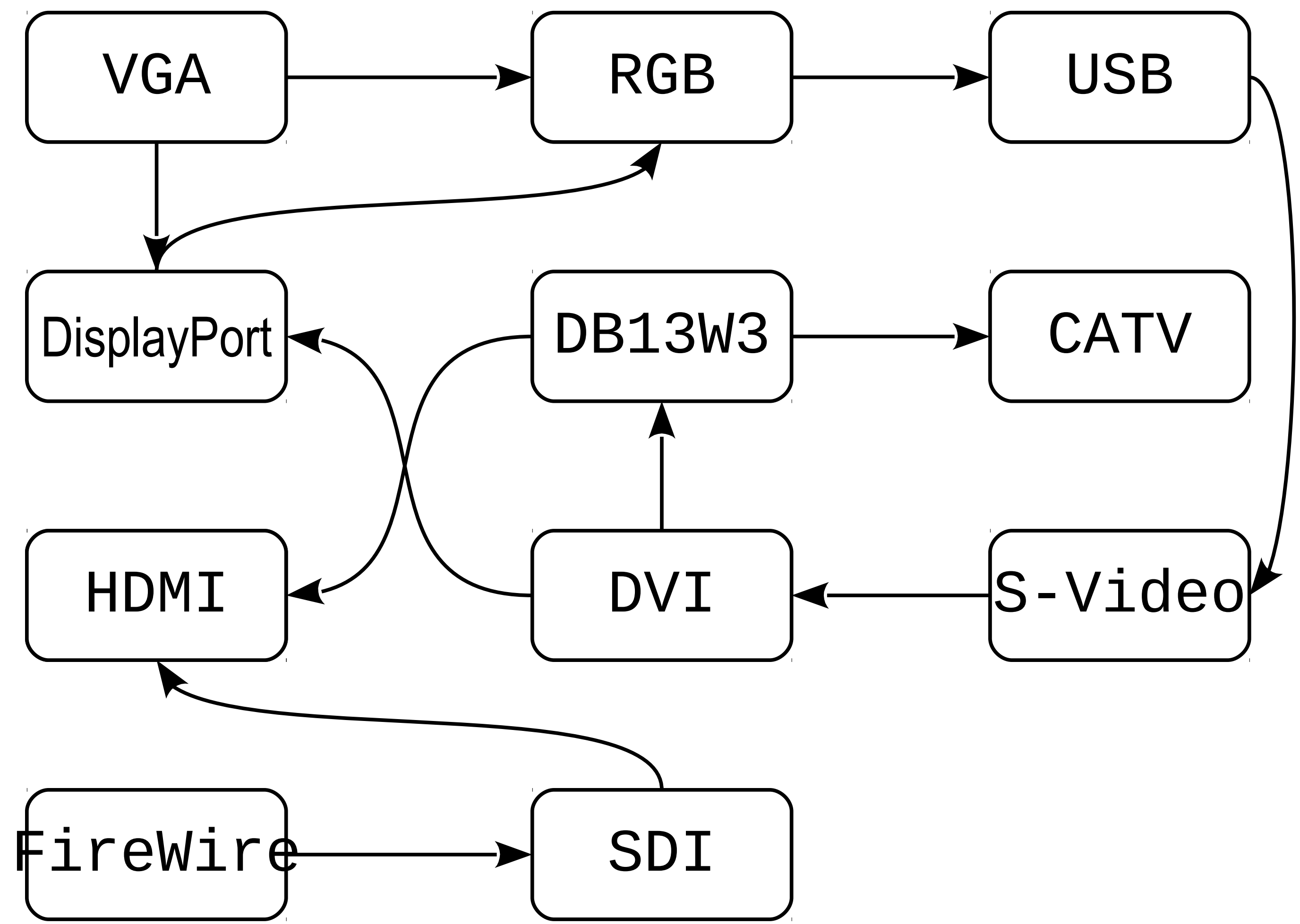

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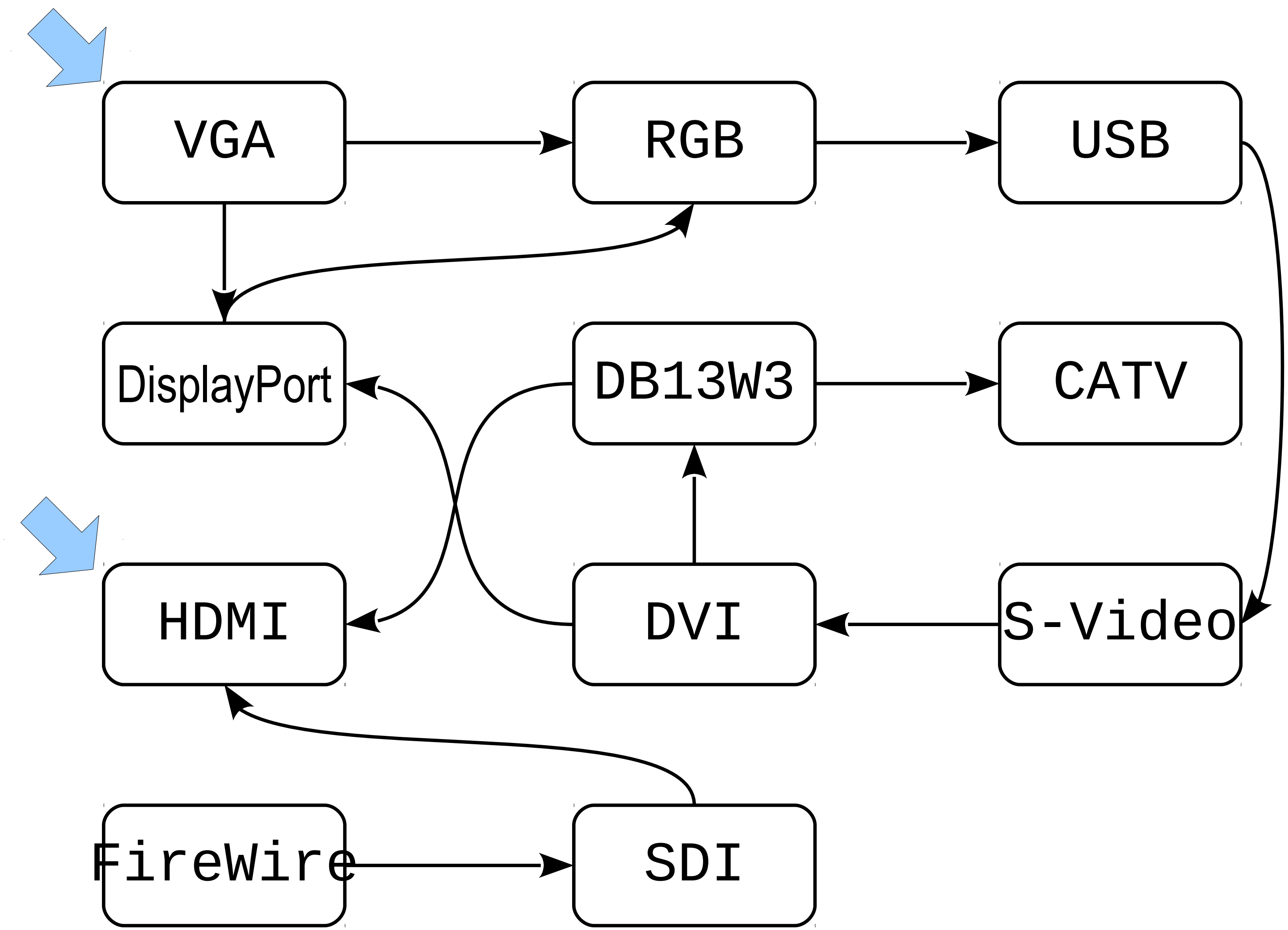

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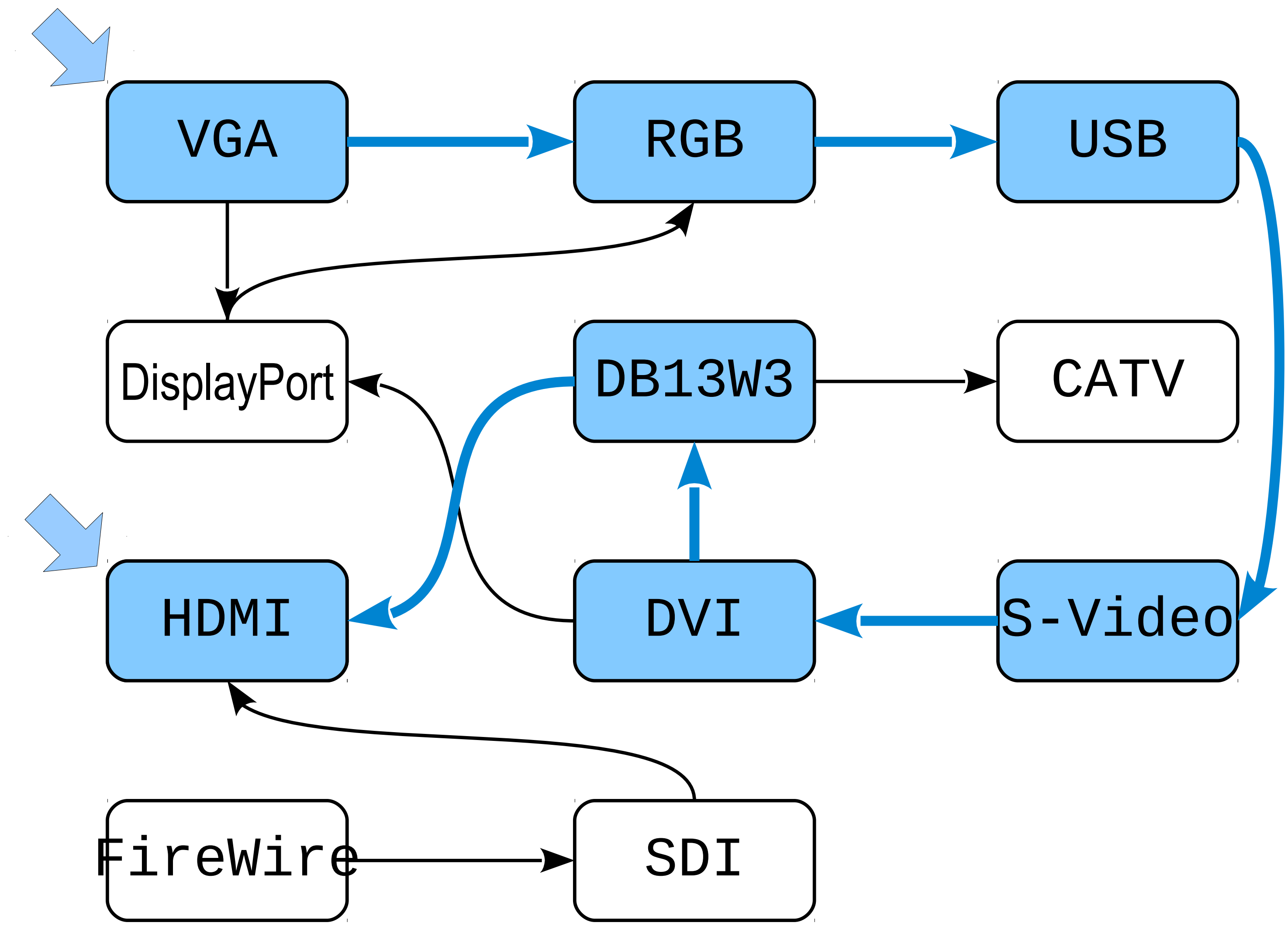

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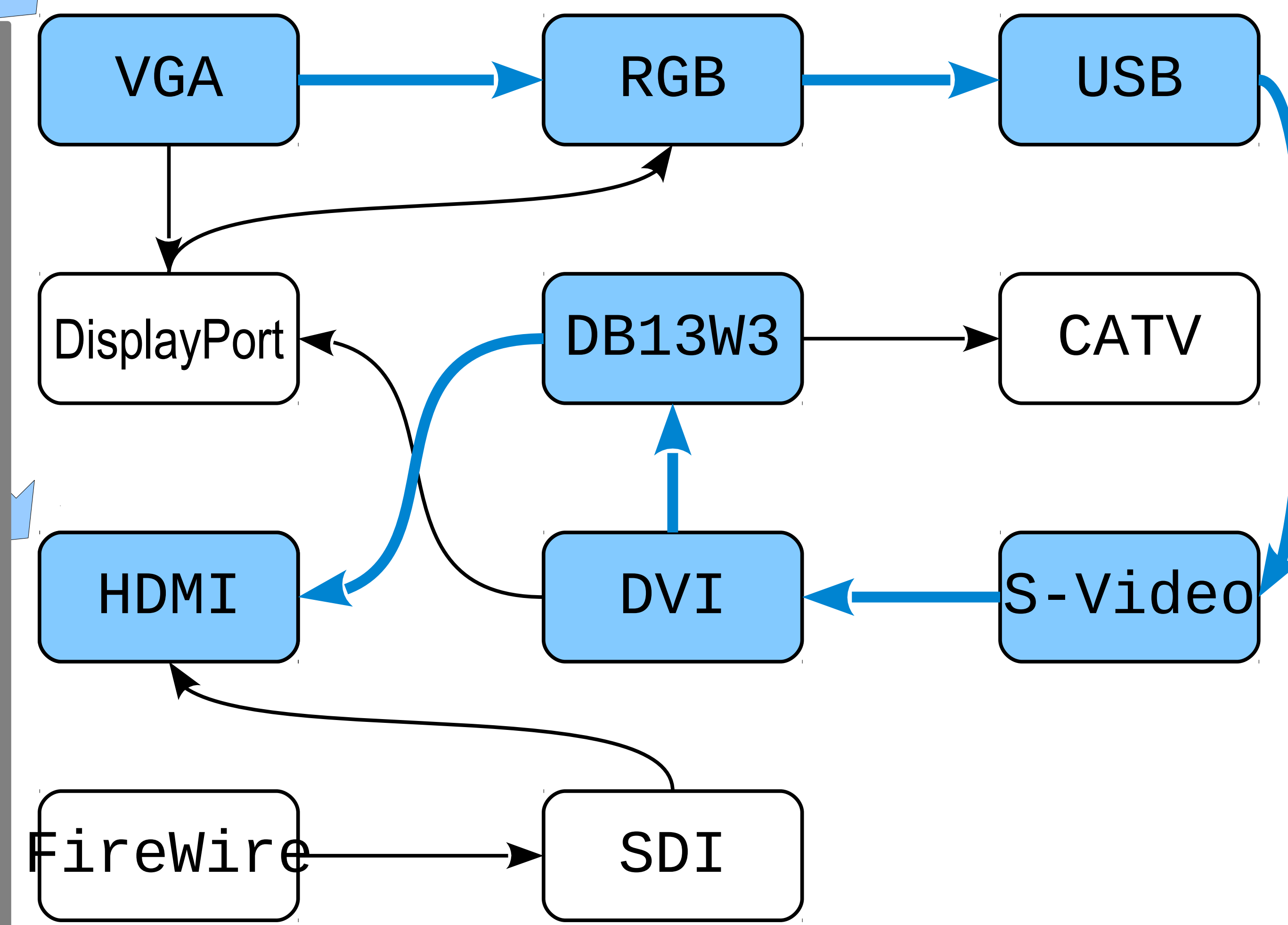

In Pseudocode

```
boolean canPlugIn(List<Plug> plugs) {
  return isReachable(plugsToGraph(plugs),
                     VGA, HDMI);
}
```

Based on this connection between plugging a laptop into a projector and determining reachability, which of the following statements would be more proper to conclude?

A. Plugging a laptop into a projector isn't any more difficult that computing reachability in a directed graph.

B. Computing reachability in a directed graph isn't any more difficult than plugging a laptop into a projector.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A** or **B**.

Intuition:

Finding a way to plug a computer into a projector can't be “harder” than determining reachability in a graph, since if we can determine reachability in a graph, we can find a way to plug a computer into a projector.

```
bool solveProblemA(string input) {
    return solveProblemB(transform(input));
}
```

Intuition:

Problem A can't be "harder" than problem B, because solving problem B lets us solve problem A.

```
bool solveProblemA(string input) {
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```

- If A and B are problems where it's possible to solve problem A using the strategy shown above*, we write

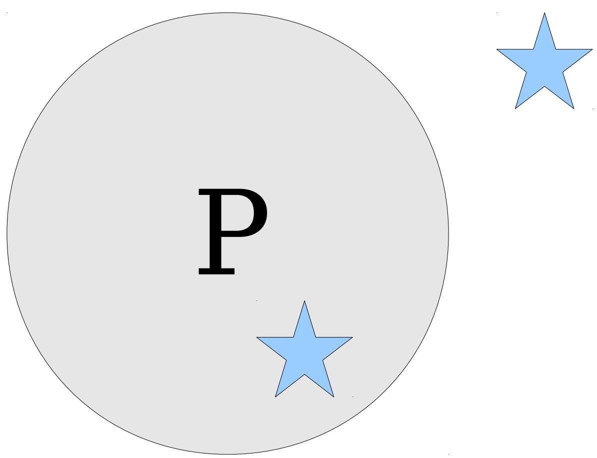
$$A \leq_p B.$$

- We say that ***A is polynomial-time reducible to B***.

* Assuming that `transform` runs in polynomial time.

Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.



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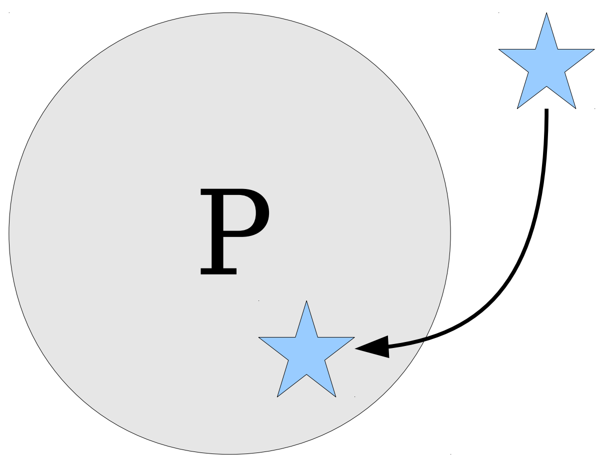

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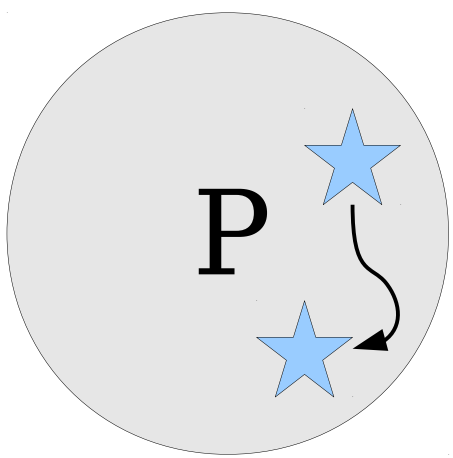

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- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.
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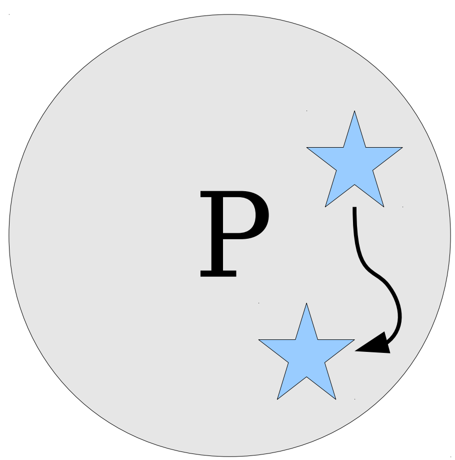

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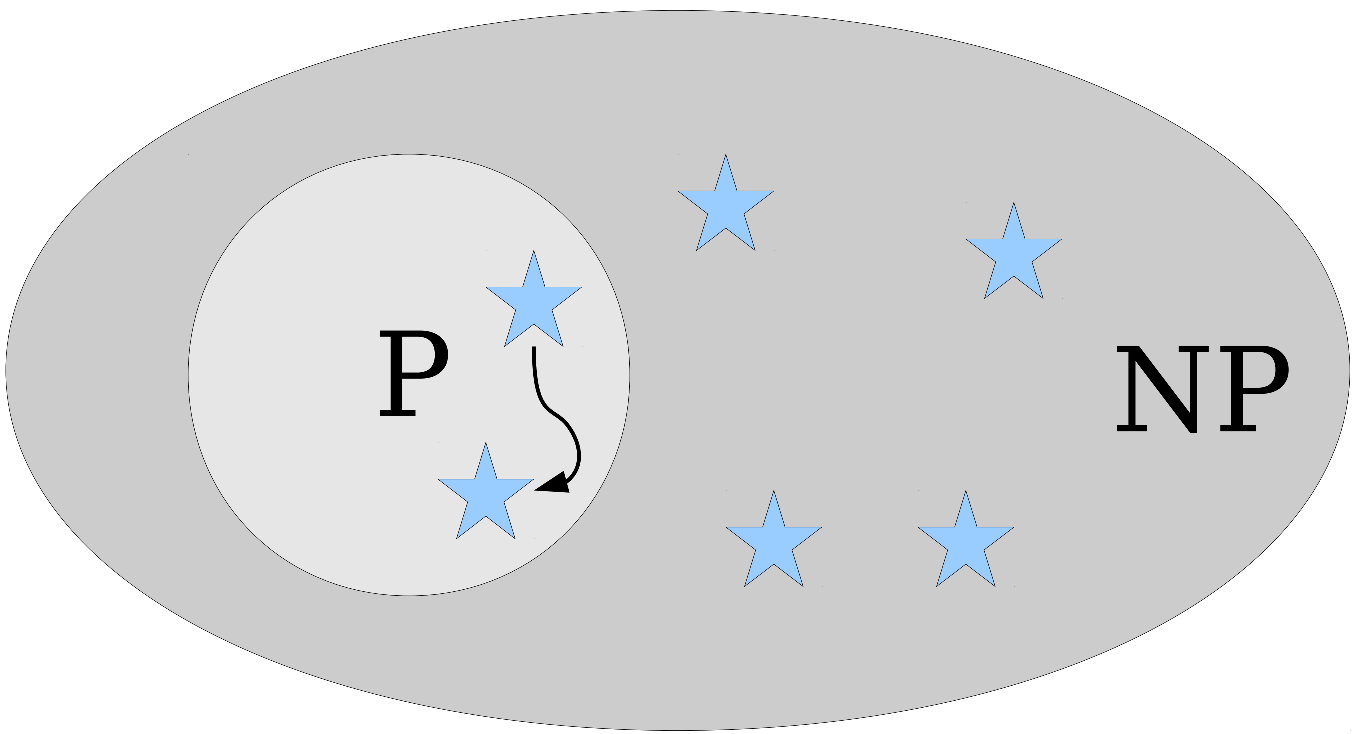

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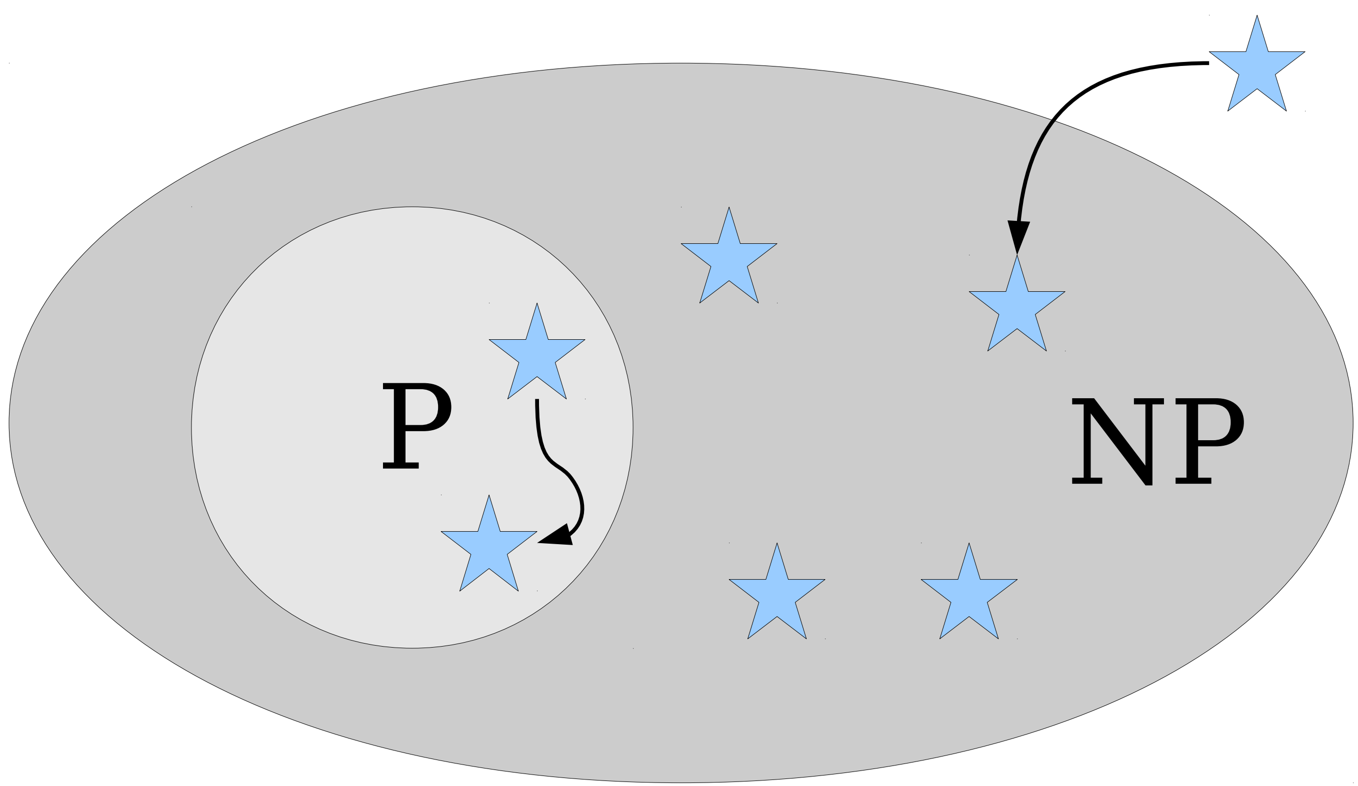

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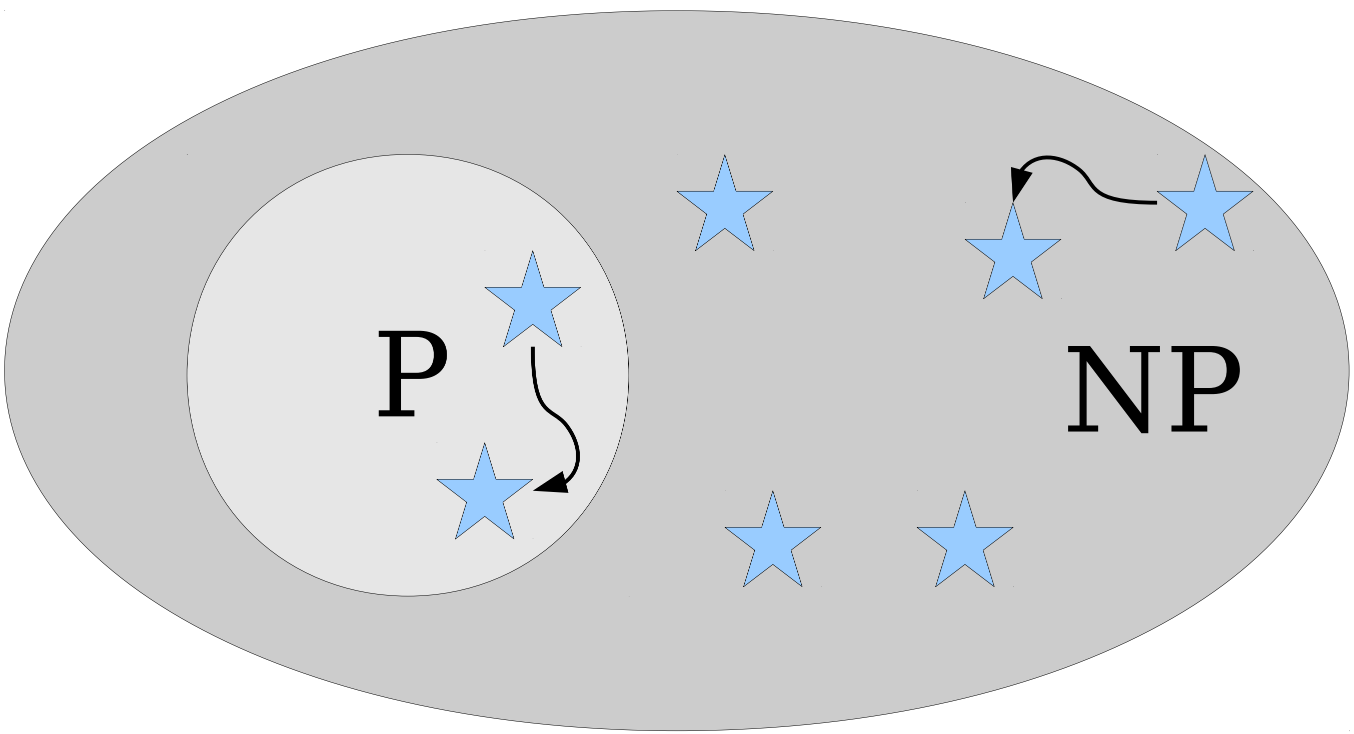


This $\leq_p$ relation lets us rank the relative difficulties of problems in **P** and **NP**.

What else can we do with it?

NP-Hardness and **NP**-Completeness

Question: What makes a problem hard to solve?

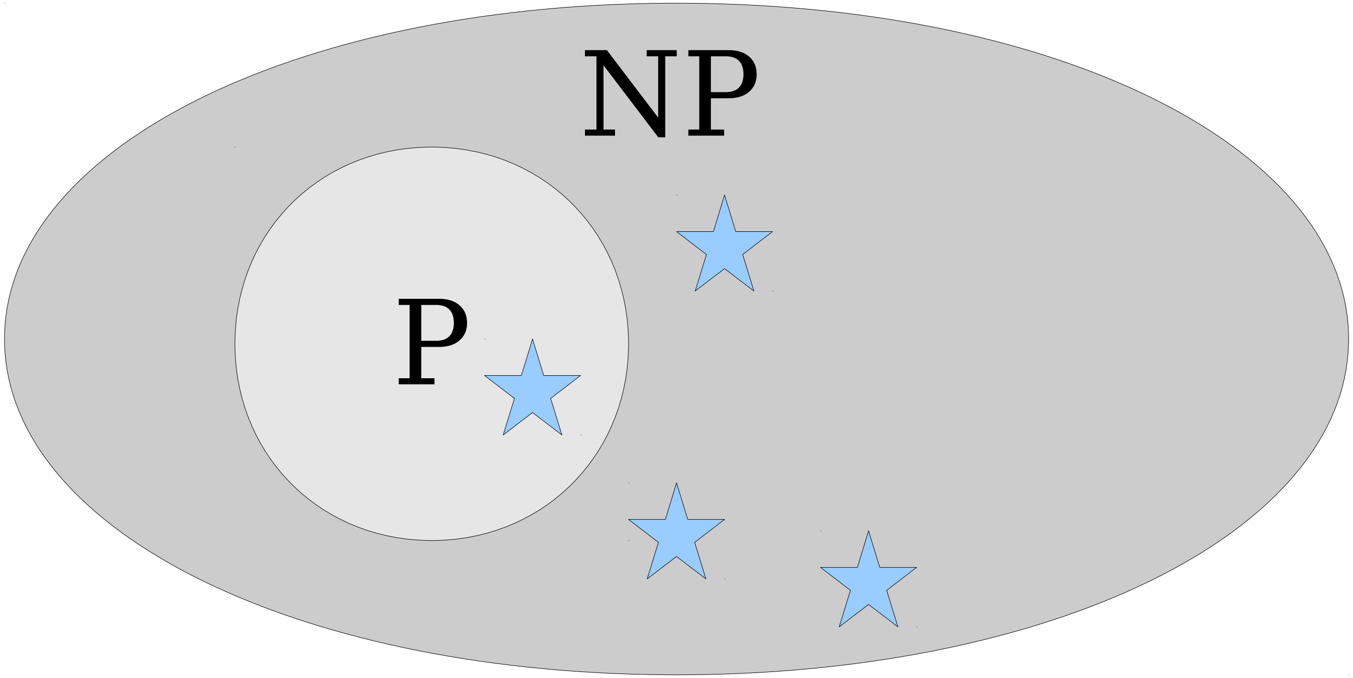
Intuition: If $A \leq_p B$, then problem B is at least as hard* as problem A.

* for some definition of “at least as hard as.”

Intuition: To show that some problem is hard, show that lots of other problems reduce to it.

NP-Hardness

- A language L is called ***NP-hard*** if for *every* $A \in$ **NP**, we have $A \leq_{\text{P}} L$.



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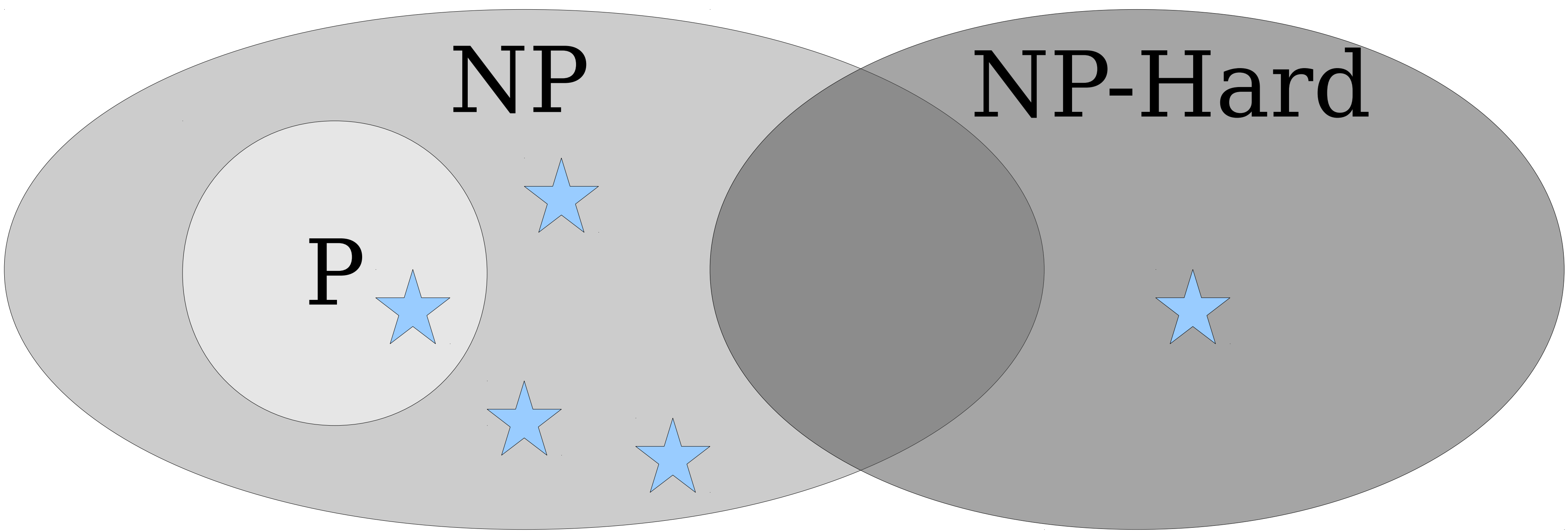

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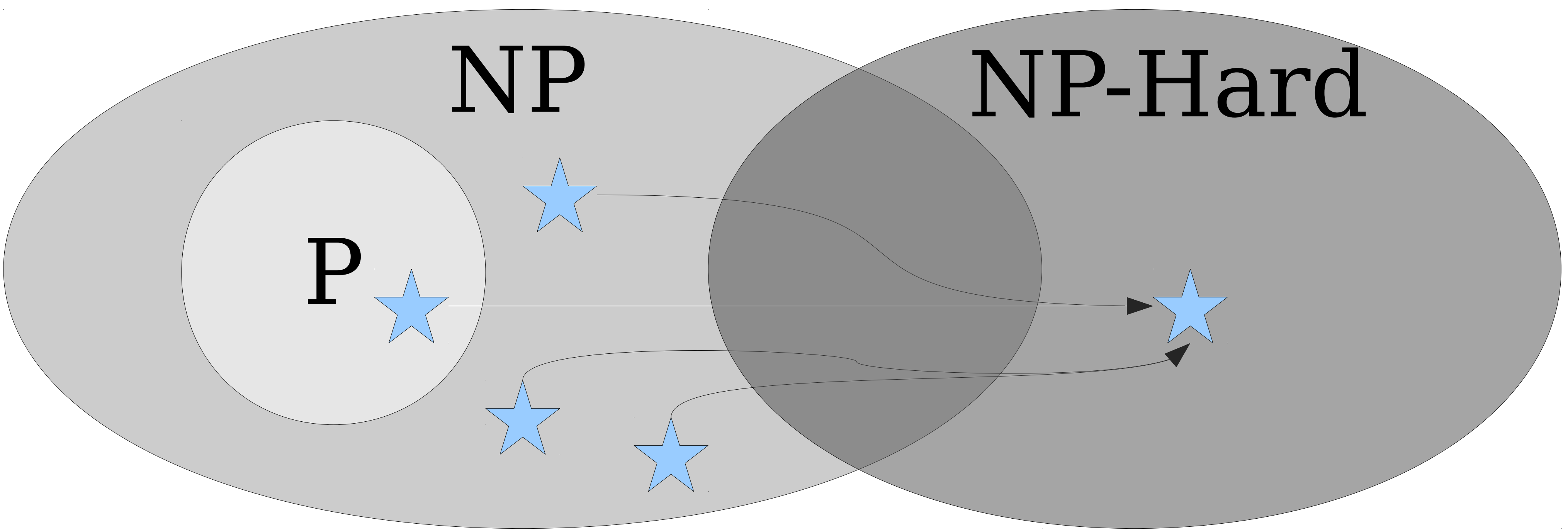

NP-Hardness

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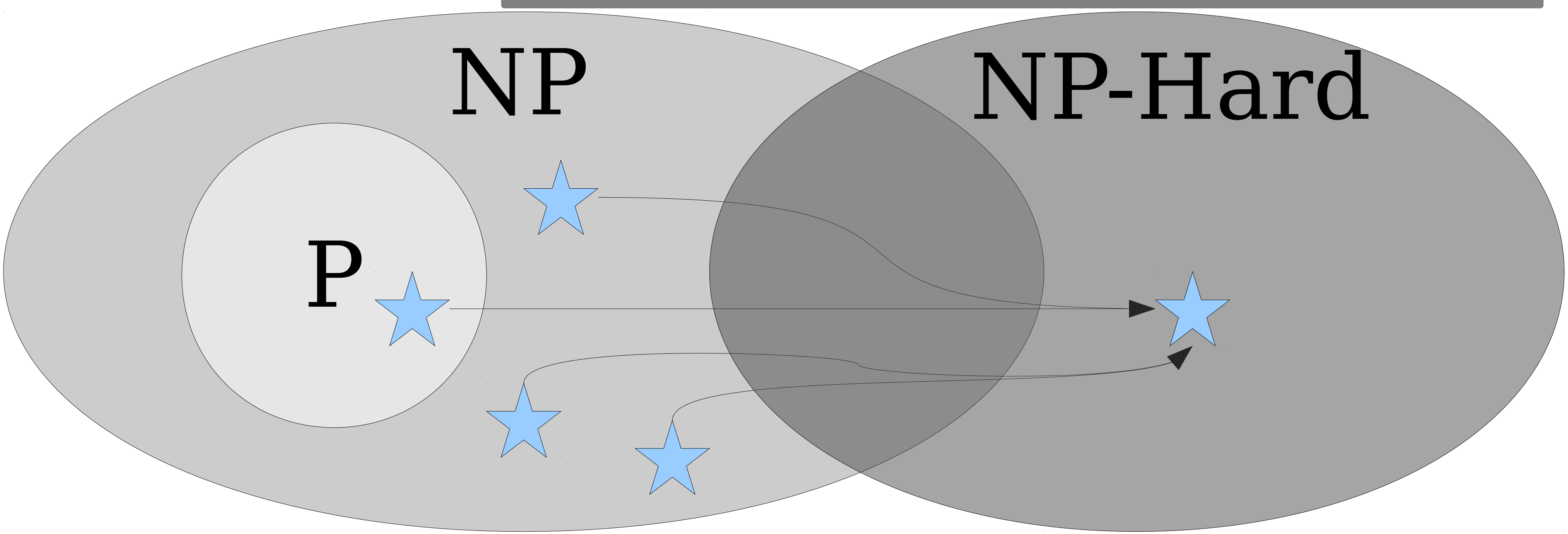
Intuitively: L has to be at least as hard as every problem in **NP**, since an algorithm for L can be used to decide all problems in **NP**.



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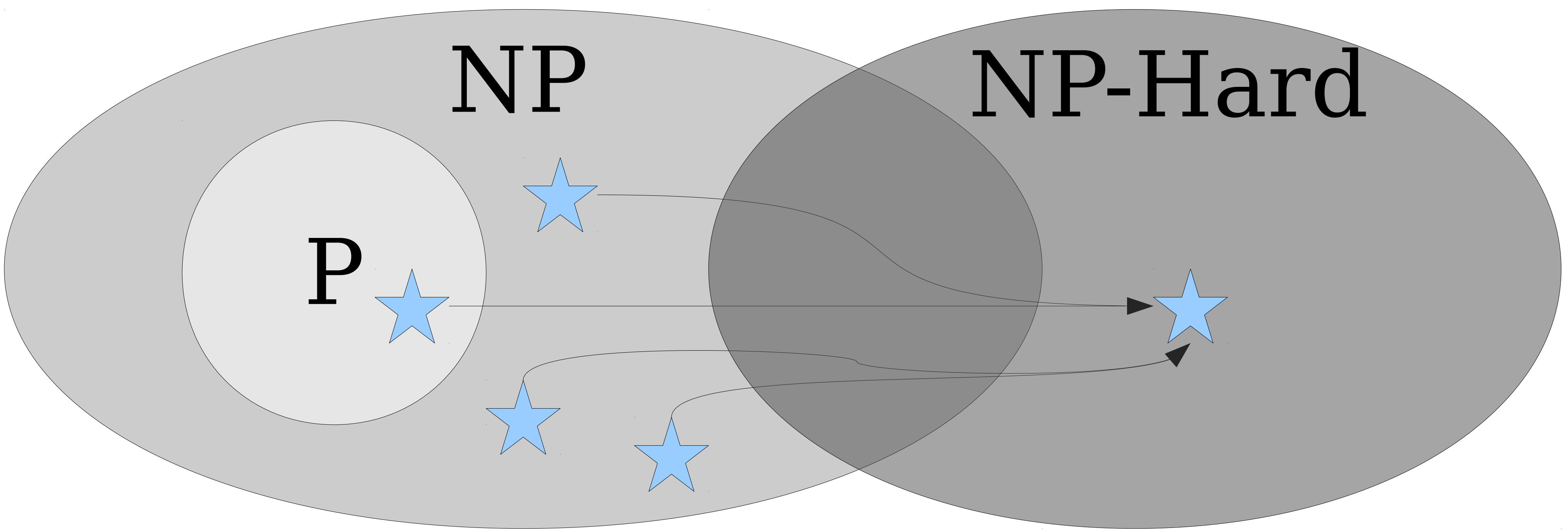

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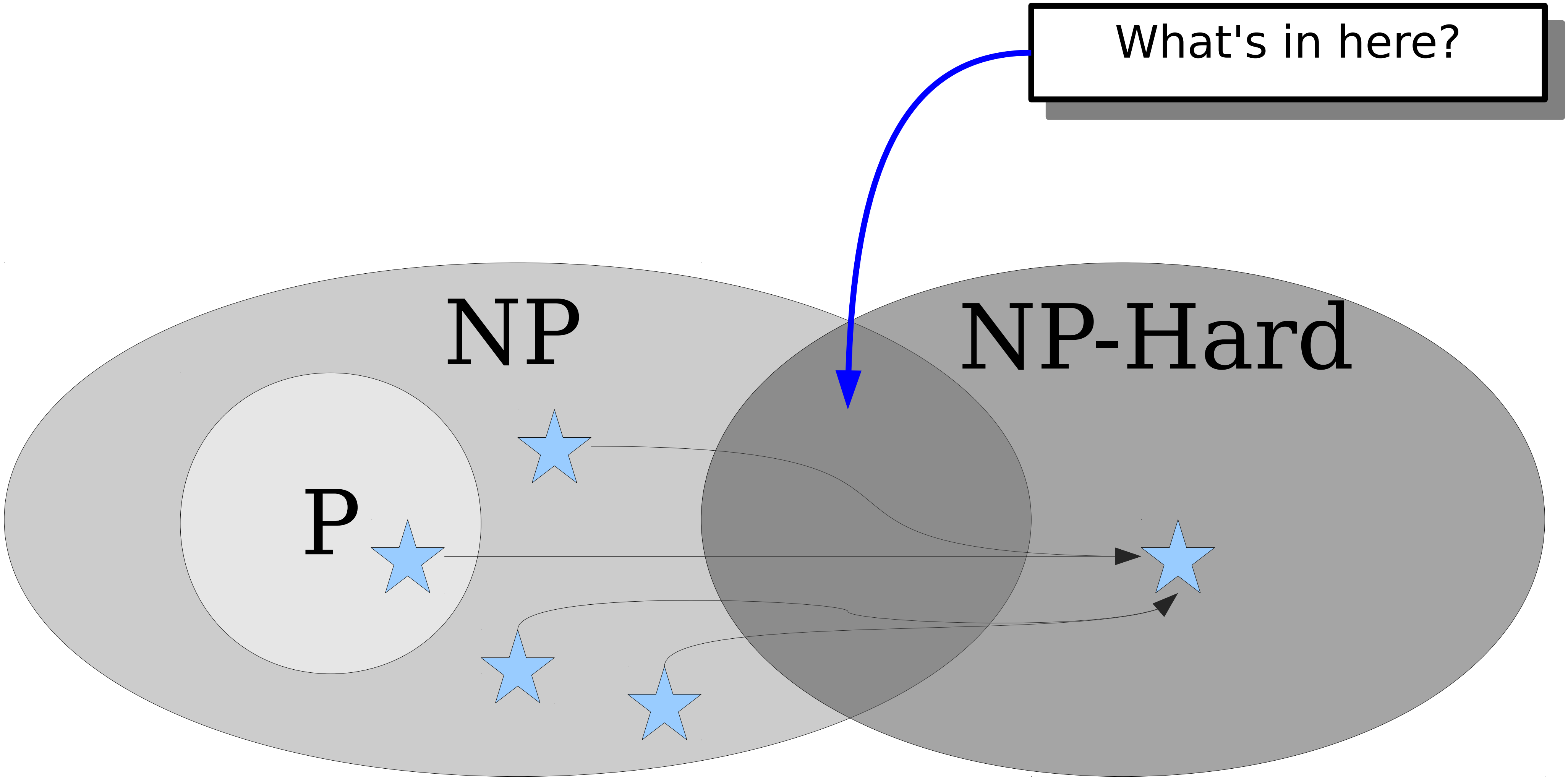
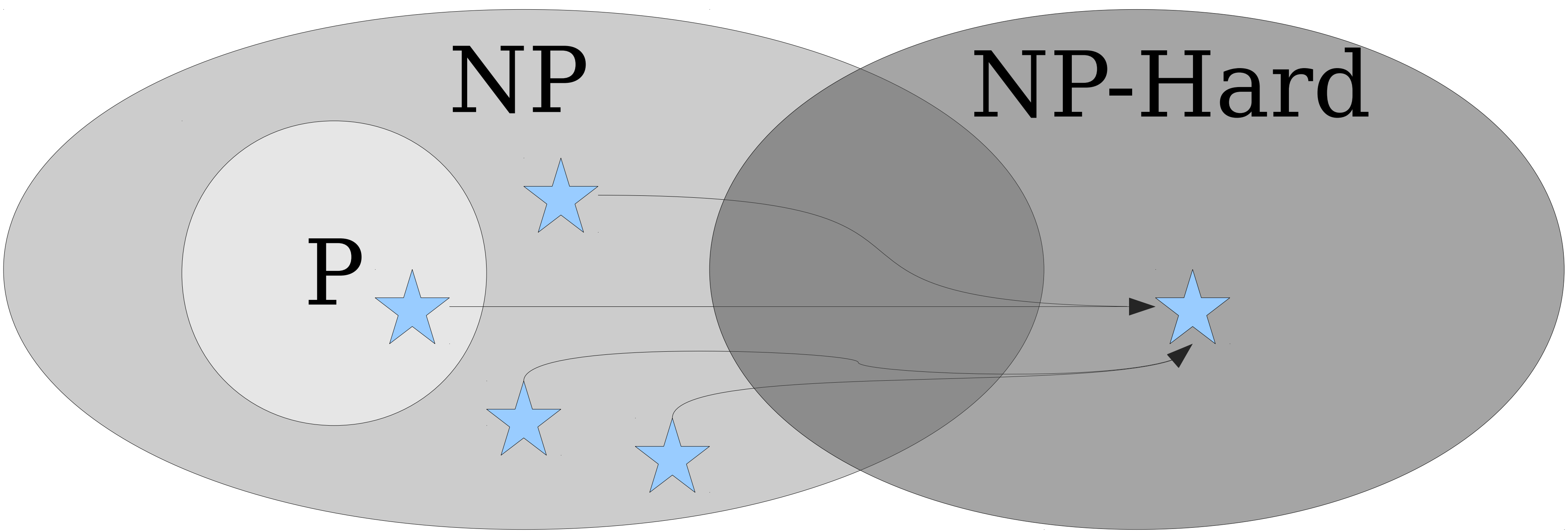


NP-Hardness

- A language L is called ***NP-hard*** if for *every* $A \in \mathbf{NP}$, we have $A \leq_P L$.
- A language in L is called ***NP-complete*** if L is **NP**-hard and $L \in \mathbf{NP}$.
- The class ***NPC*** is the set of **NP**-complete problems.



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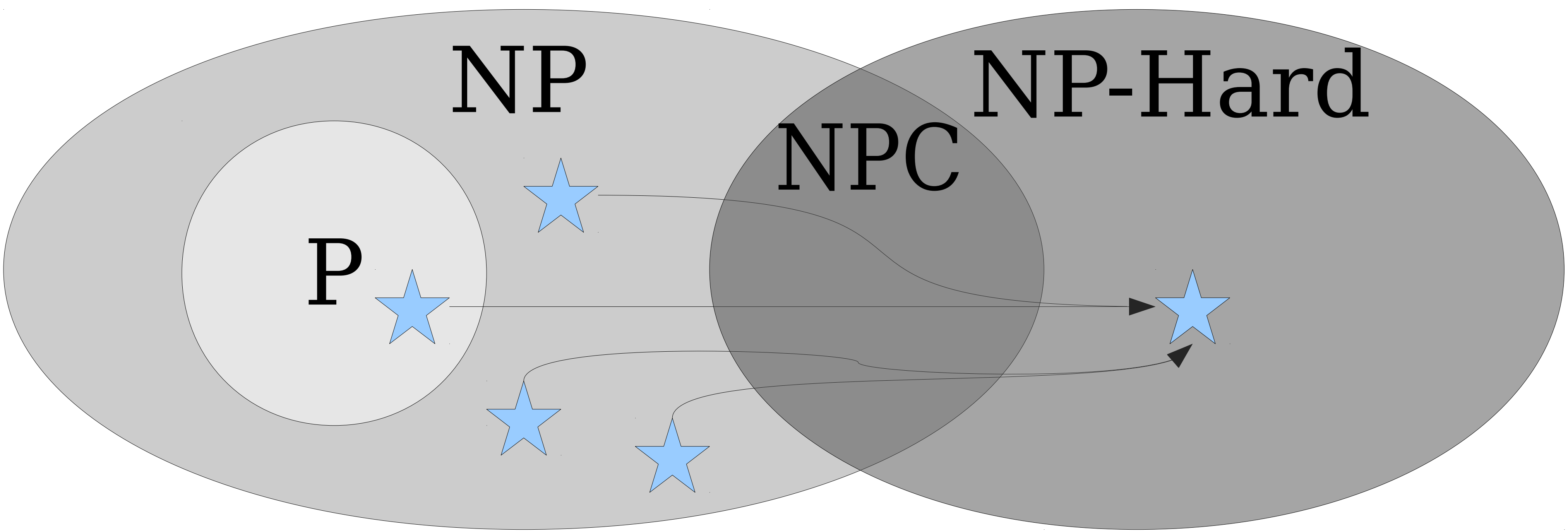

The Tantalizing Truth

Theorem: If *any* **NP**-complete language is in **P**, then **P = NP**.

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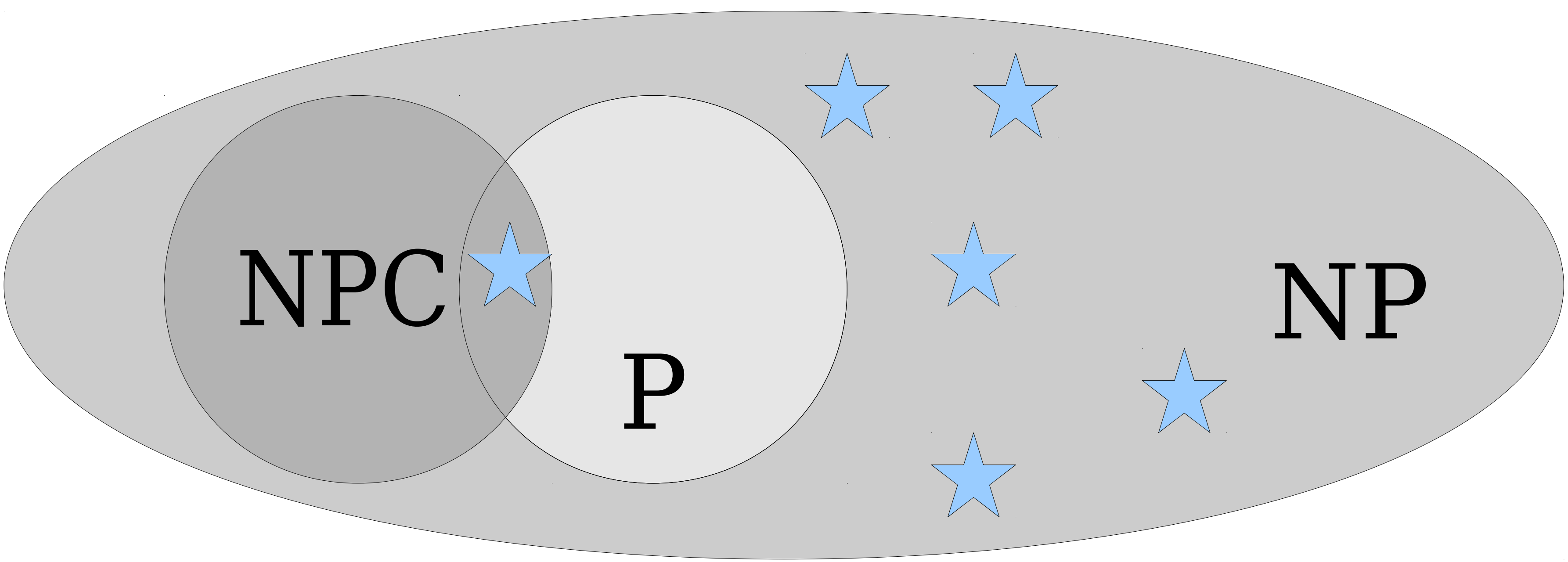

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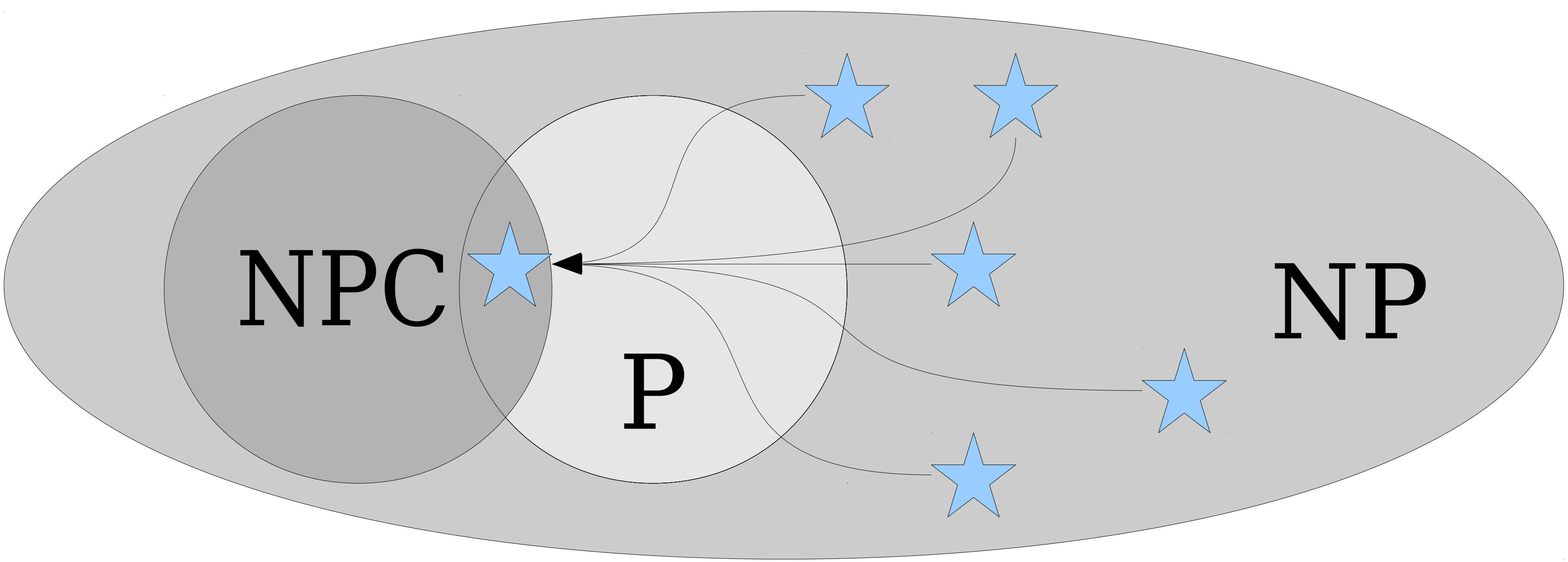

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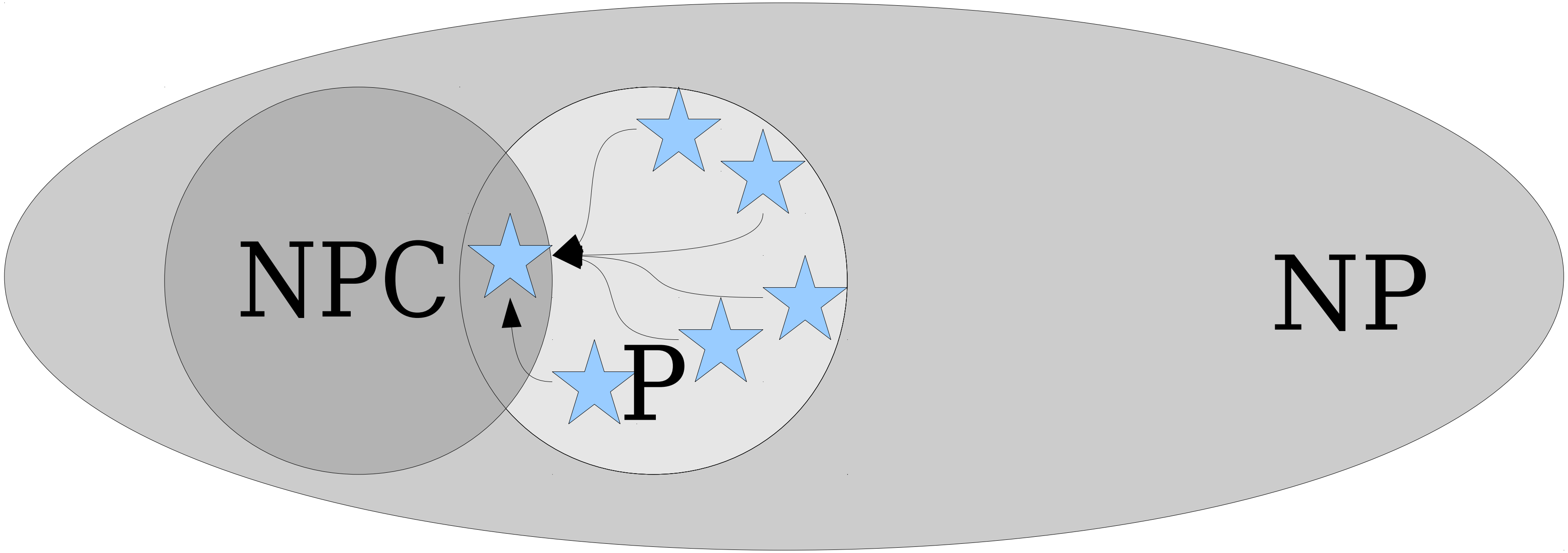

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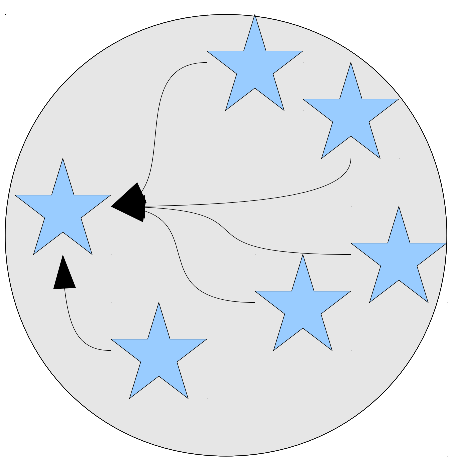
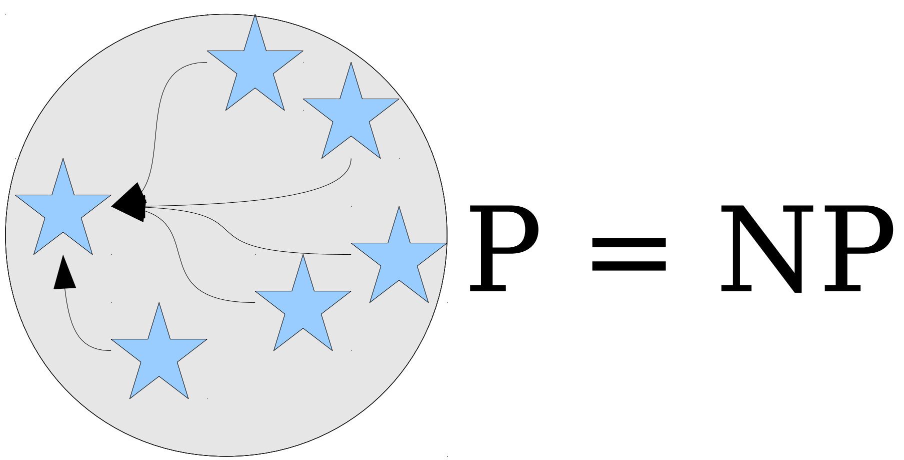


P = NP

The Tantalizing Truth

Theorem: If *any* **NP**-complete language is in **P**, then **P** = **NP**.

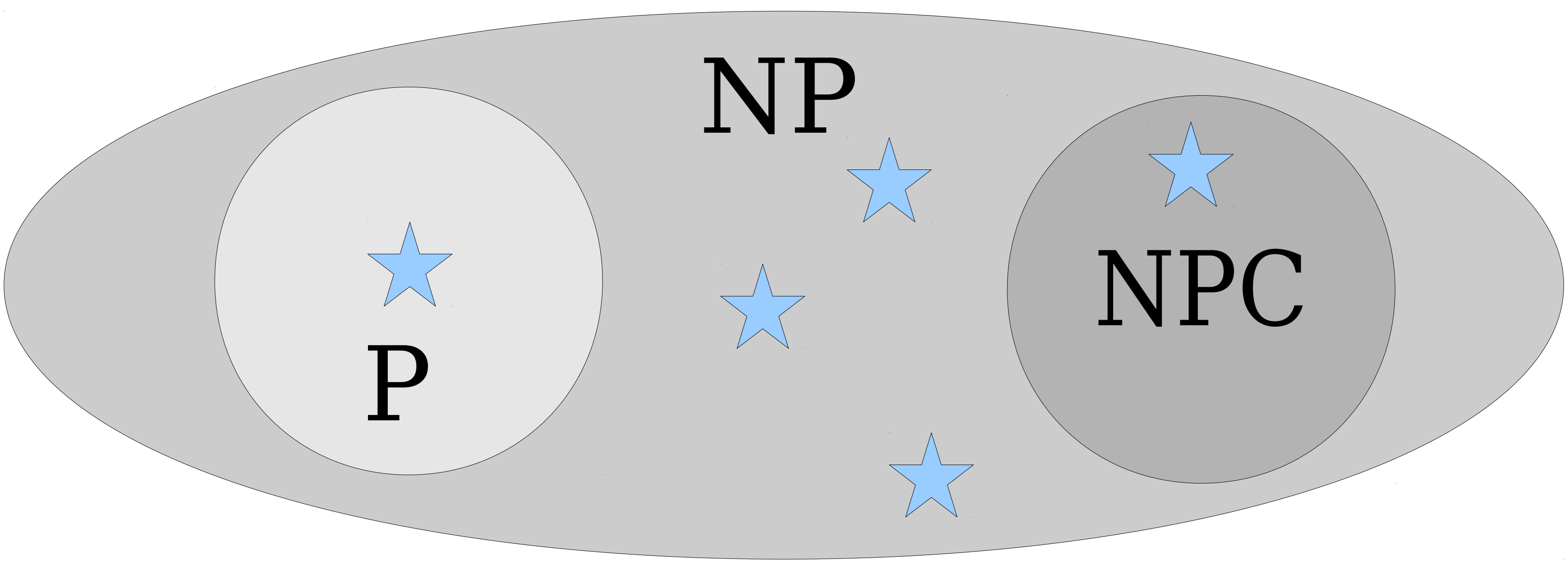
Proof: Suppose that L is **NP**-complete and $L \in$ **P**. Now consider any arbitrary **NP** problem A. Since L is **NP**-complete, we know that $A \leq_p L$. Since $L \in$ **P** and $A \leq_p L$, we see that $A \in$ **P**. Since our choice of A was arbitrary, this means that **NP** $\subseteq$ **P**, so **P** = **NP**. ■



The Tantalizing Truth

Theorem: If *any* **NP**-complete language is not in **P**, then **P** ≠ **NP**.

Proof: Suppose that L is an **NP**-complete language not in **P**. Since L is **NP**-complete, we know that $L \in$ **NP**. Therefore, we know that $L \in$ **NP** and $L \notin$ **P**, so **P** ≠ **NP**. ■



How do we even know NP-complete problems exist in the first place?

Satisfiability

- A propositional logic formula φ is called ***satisfiable*** if there is some assignment to its variables that makes it evaluate to true.
  - $p \land q$ is satisfiable.
  - $p \land \neg p$ is unsatisfiable.
  - $p \to (q \land \neg q)$ is satisfiable.
- An assignment of true and false to the variables of φ that makes it evaluate to true is called a ***satisfying assignment***.

SAT

- The ***boolean satisfiability problem*** (***SAT***) is the following:

Given a propositional logic formula φ, is φ satisfiable?

- Formally:

$$\boldsymbol{SAT = \{ \langle \varphi \rangle \mid \varphi \text{ is a satisfiable PL formula} \}}$$

SAT = { ⟨φ⟩ | φ is a satisfiable PL formula }

The language SAT happens to be in **NP**. Think about how a polynomial-time verifier for SAT might work. Which of the following would work as certificates for such a verifier, given that the input is a propositional formula φ?

A. The truth table of φ.
B. One possible variable assignment to φ.
C. A list of all possible variable assignments for φ.
D. None of the above, or two or more of the above.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then ***A***, ***B***, ***C***, or ***D***.

Theorem (Cook-Levin): SAT is **NP**-complete.

Proof Idea: To see that **SAT** ∈ **NP**, show how to make a polynomial-time verifier for it. Key idea: have the certificate be a satisfying assignment.

To show that **SAT** is **NP**-hard, given a polymomial-time verifier V for an arbitrary **NP** language L, for any string w you can construct a polynomially-sized formula $\varphi(w)$ that says "there is a certificate c where V accepts $\langle w, c \rangle$." This formula is satisfiable if and only if $w \in L$, so deciding whether the formula is satisfiable decides whether w is in L.

Proof: Take CS154!

Why All This Matters

- Resolving $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ is equivalent to just figuring out how hard SAT is.
  - If SAT $\in \mathbf{P}$, then $\mathbf{P} = \mathbf{NP}$.

    If SAT $\notin \mathbf{P}$, then $\mathbf{P} \neq \mathbf{NP}$.

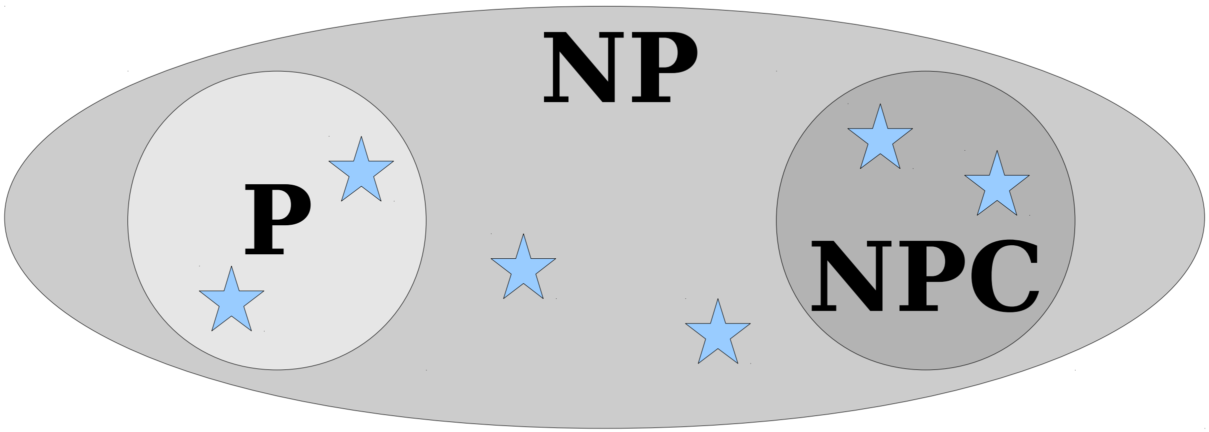
Sample **NP**-Hard Problems

- ***Computational biology:*** Given a set of genomes, what is the most probable evolutionary tree that would give rise to those genomes? *(Maximum parsimony problem)*
- ***Game theory:*** Given an arbitrary perfect-information, finite, twoplayer game, who wins? *(Generalized geography problem)*
- ***Operations research:*** Given a set of jobs and workers who can perform those tasks in parallel, can you complete all the jobs within some time bound? *(Job scheduling problem)*
- ***Machine learning:*** Given a set of data, find the simplest way of modeling the statistical patterns in that data *(Bayesian network inference problem)*
- ***Medicine:*** Given a group of people who need kidneys and a group of kidney donors, find the maximum number of people who can end up with kidneys *(Cycle cover problem)*
- ***Systems:*** Given a set of processes and a number of processors, find the optimal way to assign those tasks so that they complete as soon as possible *(Processor scheduling problem)*

Coda: What if $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ is resolved?

Intermediate Problems

- With few exceptions, every problem we've discovered in **NP** has either
  - definitely been proven to be in **P**, or
  - definitely been proven to be **NP**-complete.
- A problem that's **NP**, not in **P**, but not **NP**-complete is called ***NP-intermediate***.
- ***Theorem (Ladner):*** There are **NP**-intermediate problems if and only if **P** ≠ **NP**.



What if $\mathbf{P} \neq \mathbf{NP}$?

A Good Read:

"A Personal View of Average-Case Complexity" by Russell Impagliazzo

What if $\mathbf{P} = \mathbf{NP}$?

And a Dismal Third Option